

國立台灣科技大學一百學年度碩士班招生試題

系所組別：資訊工程系碩士班

科目：計算機數學

(總分為100分)

1. [15%]

(a) (5%) Find the matrix A of the linear transformation T from \mathbb{R}^2 to \mathbb{R}^2 with $T \begin{bmatrix} 3 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $T \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

(b) (5%) Consider two 2×2 matrices A and B . Given that $B^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$ and $(AB)^{-1} = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$, find A .

(c) (5%) If matrix A is similar to matrix B , what is the relationship between $\det(A)$ and $\det(B)$? (Show the details.)

2. [12%] Consider the matrix $A = \begin{bmatrix} \frac{1}{2} & \frac{3}{4} \\ \frac{1}{2} & \frac{1}{4} \end{bmatrix}$.

(a) (8%) Find a formula for A^k for any positive integer k .

(b) (4%) Find $\lim_{k \rightarrow \infty} A^k$.

3. [8%] Let T be an invertible linear transformation from \mathbb{R}^2 to \mathbb{R}^2 . Show that the image of the unit circle is an ellipse centered at the origin.

4. [10%] Let $\langle f(x), g(x) \rangle = \int_0^1 f(x)g(x)dx$ be the inner product defined on P_2 . Given that $S = \{1+x, 1-x\}$ is a basis of the subspace W in P_2 , find an orthonormal basis of W . (Hint: What is the space W ?)

5. [10%] Prove that every *planar* graph with 8 vertices and 13 edges cannot be 2-colored. (Hint: Start by showing there must be a triangle in such a graph.)

6. [10%] Let G be a *bipartite* graph whose vertices are divided into two sets A and B , where no two vertices in the same set are connected. Prove that if G contains a Hamiltonian path, then the number of vertices in A and the number of vertices in B differ by at most 1. (Hint: What is a Hamiltonian path?)

7. [10%] If a fair die is rolled 12 times, what is the probability that the sum of the rolls is equal to 30 exactly?

8. [12%] Let $f: \mathbb{Z} \rightarrow \mathbb{N} \cup \{0\}$ be defined by $f(x) = \begin{cases} 2x-1, & \text{if } x > 0 \\ -2x, & \text{if } x \leq 0 \end{cases}$

(a) (7%) Prove that f is one-to-one and onto.

(b) (5%) Determine f^{-1} .

9. [13%] Solve the following recurrence relations.

(a) (7%) $a_{n+2}^2 - 5a_{n+1}^2 + 6a_n^2 = 7n$, where $n \geq 0$ and $a_0 = a_1 = 1$.

(b) (6%) $a_n^2 - 2a_{n-1} = 0$, where $n \geq 1$ and $a_0 = 2$.

