※ 考生請注意：本試題不可使用計算機。 請於答案卷(卡)作答，於本試題紙上作答者，不予計分。

Show your steps clearly. Credit will be given for all the steps and derivations leading to the final results of the calculations.

1. Given that x and y are independent of each other, f is a function, f^{-1} is the inverse function of f, and f' represent taking derivative of f, write down the results of the following differentiations and integrations: **(15%)**

   (a) $\frac{d}{dx}\left(y^{f(x)}\right)$ (3%)

   (b) $\frac{d}{dy}\left(y^{f(x)}\right)$ (3%)

   (c) $\int^{y} dx \frac{1}{f'(f^{-1}(x))}$ (3%)

   (d) $\int^{y^2} dx \frac{df(x)}{dx}$ (3%)

   (e) $\frac{d}{dy}\left[\int^{y^2} dx f(x)\right]$ (3%)

2. Given $\mathbf{A} = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 8 \\ 1 & 8 & 2 \end{pmatrix}$, find $\mathbf{A}^{-1}$. **(10%)**

3. It is given that the following matrix equation is satisfied:

$$\mathbf{BX} = \mathbf{X\Lambda}, \tag{3.1}$$

where $\mathbf{B} = \begin{pmatrix} 2 & 0 & 1 \\ 8 & 2 & 8 \\ 1 & 0 & 2 \end{pmatrix}$, $\mathbf{X} \equiv \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{23} & x_{33} \end{pmatrix}$, and $\mathbf{\Lambda} \equiv \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix}$. The 9 elements of $\mathbf{X}$ and the 3 diagonal elements of $\mathbf{\Lambda}$ are to be determined under all of the following conditions: the elements d_1, d_2 and d_3 are all different numbers; and each column of $\mathbf{X}$ is of unit length, i.e.

$\sqrt{x_{1j}{}^2 + x_{2j}{}^2 + x_{3j}{}^2} = 1$ for $j = 1,2,3$.

Find a set of numbers for d_1, d_2, d_3 and all 9 elements of $\mathbf{X}$ such that Eq. (3.1) together with all of the conditions specified above are satisfied. **(15%)**

※ 考生請注意：本試題不可使用計算機。 請於答案卷(卡)作答，於本試題紙上作答者，不予計分。

4. Given $\mathbf{F}=(x^2+y^2+z^2)\hat{x}+(x^2+y^2+z^2)\hat{y}+(x^2+y^2+z^2)\hat{z}$ in Cartesian coordinates. Find the value of the surface integral $\int_S(\mathbf{F}\cdot\hat{\mathbf{n}})da$, where the surface S is a spherical surface with equation $x^2+y^2+(z-1)^2=1$ (i.e. spherical surface with center at $(x,y,z)=(0,0,1)$ and unit radius), $\hat{\mathbf{n}}$ is the unit vector normal to the surface and pointing outward, and da is the differential area. **(20%)**

5. Given the following equation in spherical coordinates (r,θ,φ)

$$\nabla^2 f=\alpha^2 f, \tag{5.1}$$

where $f=f(r)$, $\nabla^2=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$ is the Laplacian operator, and $\alpha>0$ is a real number, find the general solution of f by performing the steps described below: **(20%)**

(i) Let $f=r^\beta g$, where $g=g(r)$ and β is a constant. Convert Eq. (5.1) from an equation for f to a second-order differential equation for g in a form that explicitly consists of g' and g'', where $g'\equiv\frac{dg}{dr}$ and $g''\equiv\frac{d^2g}{dr^2}$. (5%)

(ii) Choose a value for β so as to reduce the second-order differential equation for g to a minimal number of terms. Then find the general solution for g. (7%)

(iii) Based on the result in Step (ii), find the general solution for f. (3%)

(iv) Solve for $f(r)$ under the following set of boundary conditions: $f=1$ for $r=1$ and $f\to 0$ for $r\to\infty$. (5%)

6. For a function $A(x,y,z)$, where (x,y,z) are the Cartesian coordinates, its Laplacian is

$$\nabla^2 A=\frac{\partial^2 A}{\partial x^2}+\frac{\partial^2 A}{\partial y^2}+\frac{\partial^2 A}{\partial z^2}.$$

The same function, when expressed in the cylindrical coordinates (r,φ,z), becomes $\tilde{A}(r,\varphi,z)$, i.e.

$$A(x,y,z)=\tilde{A}(r,\varphi,z).$$

While the coordinate z is the same in both coordinate systems, the other cylindrical coordinates are related to the Cartesian coordinates as follows:

$$r=\sqrt{x^2+y^2},\ \cos\varphi=\frac{x}{\sqrt{x^2+y^2}}\ \text{and}\ \sin\varphi=\frac{y}{\sqrt{x^2+y^2}}.$$

Using all of the equations given above, derive $\nabla^2\tilde{A}$ in cylindrical coordinates. **(20%)**