



(證明題, 每題 6.25 分) - 證明題須列出完整演算及邏輯過程

- Using the following *definition of right-hand limit*, prove that  $\lim_{x \rightarrow 0^+} \sqrt{x} = 0$   
 [definition of right-hand limit:  $\lim_{x \rightarrow a^+} f(x) = L$ : If for every number  $\varepsilon > 0$  there is a number  $\delta > 0$ , such that if  $0 < x - a < \delta$  then  $|f(x) - L| < \varepsilon$  ]
- $f$  and  $g$  are differentiable and  $g'(x) \neq 0$  on an open interval  $I$  that contains  $a$ . Suppose that  $\lim_{x \rightarrow a} f(x) = 0$  and  $\lim_{x \rightarrow a} g(x) = 0$ . Then,  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$ .  
 [提示:  $f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$  ]

(計算題, 每題 6.25 分 - 計算題須列出完整演算過程)

- Find  $\lim_{x \rightarrow 0} \frac{\sqrt{x^2 + 4} - 2}{x^2}$ .
- Find the second derivative ( $f''(x)$ ) of the function  $f(x) = \sqrt{x^2 + 1}$ . Express the fraction in simplest form.
- $x^2 - xy + y^3 = 3$ , find the implicit differentiation  $\frac{dy}{dx}$
- $f(x) = 2x^3 + 3x^2 + 7x + 4$ , find  $(f^{-1})'(4)$   
 [提示: Differentiable inverse function at  $a$ :  $(f^{-1})'(a) = 1/f'(f^{-1}(a))$ ]
- Differentiate the function  $y = \frac{(e^x + e^{-x})}{(e^x - e^{-x})}$ . Express the fraction in simplest form. [提示:  $(f/g)' = (gf' - fg')/g^2$ ]
- Find the limit  $\lim_{x \rightarrow -\infty} (x^2 e^x)$  [提示: L'Hospital Rule:  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$  ]
- Determine whether the series is absolutely convergent, conditionally convergent, or divergent.  

$$\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$$
- Find the Taylor series for  $f(x) = 1/\sqrt{x}$  at  $a = 9$ .
- Evaluate the integral  $\int_{-\infty}^0 x e^x dx$ .
- Evaluate the integral  $\int \frac{\sqrt{x+4}}{x} dx$ .



13. Find the solution of the initial-value problem  $x^2 \frac{dy}{dx} + xy = 1 \quad x > 0, \quad y(1) = 2$

14. (證明題) Show that the function  $u(x, y) = e^x \sin y$  is a solution of Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

15. Find the directional derivative of the function  $f(x, y) = x^2 y^3 - 4y$  at the point  $(2, -1)$  in the direction of the vector  $\mathbf{v} = 2\mathbf{i} + 5\mathbf{j}$ .

16. Evaluate the double integral  $\iint_R (x - 3y^2) dA$ , where  $R = \{(x, y) \mid 0 \leq x \leq 2, 1 \leq y \leq 2\}$ .