國立高雄大學 107 學年度研究所碩士班招生考試試題

系所：應用數學系

科目：線性代數　　身份別：一般生應用數學組、在職生應用數學組　　是否使用計算機：否

考試時間：100 分鐘　　本科原始成績：100 分

Notations.

$P_n(R)$: the set of all polynomials of degree at most n with coefficients from R.
$M_{m\times n}(R)$: the set of all $m \times n$ matrices with entries from R.
$N(T)$: the null space of T.
$\det(A)$: the determinant of the matrix A.
$W_1 + W_2 = \{u + v : u \in W_1 \text{ and } v \in W_2\}$.

Part I (15) Label the following statements as True (T) or False (F).

1. The empty set is linearly dependent.
2. If $T, U : R^2 \to R$ are both linear and $T(1,1) = U(1,1)$ and $T(1,-1) = U(1,-1)$, then $T = U$.
3. The function $T : M_{3\times 3}(R) \to R$ defined by $T(A) = \det(A)$ is a linear transformation.
4. For every matrix $A \in M_{6\times 6}(R)$, $\det(-A) = -\det(A)$.
5. Similar matrices always have the same eigenvalues.

Part II Answer the following questions. To get full credit you must show all work!

1. $W_1 = \{(a, 0) : a \in R\}$ and $W_2 = \{(0, a) : a \in R\}$ are two subspaces of R^2.
   (a) (8) Is $W_1 \cup W_2$ a subspace of R^2?
   (b) (7) Find $W_1 + W_2$.
2. (10) Find all scalars a, if any exist, such that $x^2 + 1$, $2x^2 + ax + 3$, and $2x^2 + 3x + 1$ are linearly independent vectors in $P_2(R)$.
3. (10) Let T be the linear operator on R^2 that rotates each vector in the plane through an angle of $\pi/2$. Determine whether T is diagonalizable or not.
4. (10) Let V and W be vector spaces, and let $T : V \to W$ be a linear transformation. Show that T is one-to-one if and only if $N(T) = \{0\}$.
5. (10) Let A be an $n \times n$ matrix that is similar to an upper triangular matrix and has the distinct eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_k$ with corresponding multiplicities $m_1, m_2, \cdots, m_k$. Find $\det(A)$.
6. (15) Let $A = \begin{pmatrix} k & 1 & 1 & 1 \\ 1 & k & 1 & 1 \\ 1 & 1 & k & 1 \\ 1 & 1 & 1 & k \end{pmatrix}$. Find $\text{rank}(A)$.
7. (15) Find the general solution to the system of differential equations below

$$x' = 8x + 10y,$$
$$y' = -5x - 7y.$$