Problem 1.

(10%)

(a) Solve the differential equation,

$$\frac{dy}{dx} = \frac{4x \cdot \sin y + y \cdot \cos x}{3y^2 - 2x^2 \cdot \cos y - \sin x}.$$

(15%)

(b) Solve the following differential equation, using Laplace transformation method. $y(t)$ is a function of t, and $u(t)$ is the unit step function.

$$y'' + 2y = u(t - \frac{\pi}{\sqrt{2}}) - u(t - \sqrt{2}\pi),\ y(0) = 1 \text{ and } y'(0) = 0$$

Problem 2.

(15%)

(a) Please find the eigenvalues and eigenvectors of the matrix,

$$\mathbf{A} = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}.$$

(10%)

(b) Given two vectors:

$$\mathbf{V}_1 = \begin{bmatrix} 4 \\ -8 \\ 1 \\ 0 \end{bmatrix} \text{ and } \mathbf{V}_2 = \begin{bmatrix} 20 \\ 0 \\ k \\ 1 \end{bmatrix}.$$

For what value of k are the vectors $\mathbf{V}_1$ and $\mathbf{V}_2$ orthogonal?

Problem 3.

(10%)

(a) Check if the vector $[8 \quad -6 \quad -5 \quad 17]$ is a linear combination of $[2 \quad -1 \quad 1 \quad 3]$,

$[3 \quad 1 \quad 4 \quad -1]$, and $[8 \quad -1 \quad 0 \quad 6]$.

(15%)

(b) Solve the initial value problem.

$$x_1' = 4x_1 - 6x_2$$
$$x_2' = x_1 - x_2$$
$$x_1(0) = 1,\ x_2(0) = 1$$

Problem 4.

(10%)

(a) Find $u(x, y)$ of the partial differential equation satisfying the given condition.

$$3\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0;\ u(1,3) = 7$$

(15%)

(b) Derive a Fourier series solution $y(x, t)$ of the boundary value problem.

$$\frac{\partial^2 y}{\partial t^2} = 4\frac{\partial^2 y}{\partial x^2}\ (0 < x < 5,\ t > 0)$$

$$y(0,\ t) = y(5,\ t) = 0\ (t > 0)$$

$$y(x,\ 0) = 0\ (0 < x < 5)$$

$$\frac{\partial y}{\partial t}(x,\ 0) = x\ (0 < x < 5)$$