國立彰化師範大學 100 學年度碩士班招生考試試題

系所：物理學系　　組別：甲組　　科目：物理數學

☆☆請在答案紙上作答☆☆　　　　共 2 頁，第 1 頁

1. (18%) Given a force $\vec{F}=(2y,x,z^2)$ in the Cartesian coordinates, evaluate the work done by it from (0,0,0) to (1,1,1) along the following curves: (a) the rectilinear path from (0,0,0) to (1,0,0) to (1,1,0) to (1,1,1); (b) the curve which is the intersection of the paraboloid $x^2+y^2=2z$ and the plane $x=y$.

2. The amplitude $A(\vec{r})$ of a scalar wave satisfies the wave equation

$$\nabla^2 A=\frac{1}{v^2}\frac{\partial^2 A}{\partial t^2} \quad ,$$

where v is the wave speed. Suppose that $A(\vec{r})$ depends only on r and t, i.e. $A=A(r,t)$, where $r=|\vec{r}|$.

(a) (9%) Show that the wave equation can be written as

$$\frac{\partial^2}{\partial r^2}(rA)=\frac{1}{v^2}\frac{\partial^2}{\partial t^2}(rA)$$

(b) (9%) Show that if $A(r,0)=0$ and $\partial A/\partial t\,(r,0)=0$, then

$$\left(\frac{d^2}{dr^2}-k^2\right)r\overline{A}=0 \quad ,$$

where $\overline{A}(r,s)$ is the time Laplace transform of $A(r,t)$ and $k=\sqrt{s/v}$.

(c) (14%) In terms of the solution to the differential equation in (b), show that the solution of the wave equation satisfying the initial conditions given in (b) is of the form

$$A(r,t)=\frac{f\left(t-r/v\right)}{r} \quad ,$$

provided that $f(t)=0$ for $t<0$.

3. A light string of length $3a_0$ is stretched between two fixed points a distance $3a$ apart $(a>a_0)$. Two particles of mass m are attached so as to divide the string into three equal sections. The system rests on a smooth horizontal plane and the particles can perform longitudinal horizontal oscillations. Assuming that the displacements of the two particles are small (compared with a),

(a) (5%) show that the equations of motion for $x_1(t)$ and $x_2(t)$, the displacements of the particles respectively, are

$$\ddot{x}_1 + n^2(2x_1 - x_2) = 0$$
$$\ddot{x}_2 + n^2(-x_1 + 2x_2) = 0$$

where $n^2 = k/m$ and the tension is k times the extension.

(b) (10%) Find the eigen-frequencies of this system and their corresponding eigenvectors.

(c) (10%) Write down the solutions corresponding to each normal mode and give a sketch showing how the system vibrates in each case.

4. The Legendre polynomials, $\{P_n(x)\}$, are defined on the interval $-1 \le x \le 1$ via a generating function $g(x,t)$ by the relation

$$g(x,t) = (1 - 2xt + t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x) t^n .$$

(a) (5%) If a function $f(x)$ can be Legendre-expanded as

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} P_n(x)$$

within the interval $-1 \le x \le 1$. What is the function $f(x)$?

(b) (10%) Find the values of the integral

$$\int_{-1}^{1} x^2 P_n(x) dx .$$

5. (10%) Find the value of the integral

$$I = \int_0^{\infty} \frac{dx}{(x+4)^3 x^{1/2}} .$$