國立彰化師範大學 100 學年度碩士班招生考試試題

系所：數學系　　　　　　　　　　　　　　科目：線性代數

☆☆請在答案紙上作答☆☆　　　　　　　　　　共 1 頁，第 1 頁

1. Let $W=\left\{\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}: x_1+x_2+x_3+x_4=0\right\}$ be a subspace of $\mathbb{R}^4$.

(1) Find a basis of W. (7%)

(2) Find an orthonormal basis of W. (8%)

2. Let V be a real vector space and let W_1 and W_2 be two subspaces of V. Prove that if $W_1 \cup W_2$ is a subspace of V, then $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$. (15%)

3. Let V be the real vector space of real valued functions defined on $(0,\infty)$.

(1) Is $\{\cos x, \sin x, e^x\}$ a linearly independent set in V? Explain. (6%)

(2) Is $\{\cos x, \cos 2x, \cos 3x, \cos^2 x, \cos^3 x\}$ a linearly independent set in V? Explain. (6%)

(3) Is $\left\{\frac{1}{(x+1)}, \frac{1}{(x+1)^2}, \dots, \frac{1}{(x+1)^n}\right\}$ a linearly independent set in V? Explain. (8%)

4. Let $A=\begin{bmatrix}1 & 0 & 3\\ 2 & -1 & 3\\ 0 & 0 & -2\end{bmatrix}$. Compute A^{100}. (17%)

5. (1) Let $\mathbf{e}_0=[0,0]^T$, $\mathbf{e}_1=[1,0]^T$ and $\mathbf{e}_2=[0,1]^T$. Suppose $\mathbf{v}_0,\mathbf{v}_1,\mathbf{v}_2 \in \mathbb{R}^2$ are vertices of a triangle. Show that there exist a unique invertible 2×2 matrix A and a unique column vector $\mathbf{b}\in\mathbb{R}^2$ so that the map $f:\mathbb{R}^2\to\mathbb{R}^2$, given by $f(\mathbf{x})=A\mathbf{x}+\mathbf{b}$ for $\mathbf{x}\in\mathbb{R}^2$, satisfies $f(\mathbf{e}_i)=\mathbf{v}_i$, $i=0,1,2$. (8%)

(2) Suppose $\mathbf{u}_0,\mathbf{u}_1,\mathbf{u}_2\in\mathbb{R}^2$ are vertices of a triangle. Prove (1) with $\mathbf{e}_i$ replaced by $\mathbf{u}_i$. (8%)

6. Let $M_2(\mathbb{R})$ be the real vector space of all 2×2 matrices with real coefficients. For a 2×2 matrix A, define $T_A: M_2(\mathbb{R})\to M_2(\mathbb{R})$ by $T_A(X)=AX-XA$. Show that if A is not a multiple of the identity matrix, then the rank of $\operatorname{rank}(T_A)$ is 2 [Hint: Determine the rank of T_A if A is a Jordan matrix]. (17%)