國立成功大學 107 學年度碩士班招生考試試題

系　所：統計學系

考試科目：數理統計

考試日期：0206，節次：2

※ 考生請注意：本試題不可使用計算機。　請於答案卷(卡)作答，於本試題紙上作答者，不予計分。

1. (10%) Suppose we generate random variables X and Y by the following algorithm:

$$f_Y(y) = \begin{cases} 2e^{-2y}, & 0 < y < \infty; \\ 0 & \text{elsewhere.} \end{cases}$$

$$f_{X|Y}(x|y) = \begin{cases} e^{-(x-y)}, & y < x < \infty; \\ 0 & \text{elsewhere.} \end{cases}$$

What is the probability density function of X generated by the above algorithm?

2. (10%) Suppose that an engineer wishes to compare the number of complaints per week filed by union stewards for two different shifts at a manufacturing plant. One hundred independent observations on the number of complaints gave mean $\bar{x} = 20$ for shift 1 and $\bar{y} = 22$ for shift 2. Assume that the number of complaints per week on the ith shift has a Poisson distribution with mean θ_i, for $i = 1, 2$. Use the likelihood ratio method to test $H_0: \theta_1 = \theta_2$ versus $H_0: \theta_1 \neq \theta_2$ with $\alpha = 0.01$.

3. (10%) Assume that a random variable has the uniform density

$$f(x) = \begin{cases} \frac{1}{\theta}, & \text{for } 0 < x < \theta; \\ 0, & \text{elsewhere.} \end{cases}$$

We want to estimate parameter θ on the basis of a single observation. If the decision function is to be of the form $d(x) = kx$, where $k \geq 1$, and the losses are proportional to the absolute value of the errors, that is

$$L(kx, \theta) = c|kx - \theta|,$$

where c is a positive constant. Find the value of k that will minimize the risk and what would the estimate of θ be when we observed $x = 5$?

4. Let $(x_1, Y_1), \ldots, (x_n, Y_n)$ be n pairs of independent samples. Consider the Poisson regression model with the probability mass function denoted as $Y_i \sim \text{Poisson}(x_i\beta)$,

$$P(Y = y|x, \beta) = \frac{(x\beta)^y e^{-x\beta}}{y!}, y = 0, 1, 2, \ldots,$$

where $i = 1, \ldots n$ and $x_1, \ldots, x_1$ are positive known constants.

a) (5%) Find the maximum likelihood estimator (MLE) for β, denoted as $\hat{\beta}$.
b) (5%) Compute the mean and variance of $\hat{\beta}$.
c) (5%) Assume that β has the gamma prior density

$$P(\beta|a, b) = \frac{a^{ab}}{\Gamma(ab)} \beta^{ab-1} e^{-a\beta},$$

where $a > 0$, $b > 0$, and $E(\beta) = b$. Find the posterior density of β given $\boldsymbol{Y} = (Y_1, \ldots, Y_n)$.
d) (10%) Compute the posterior mean of β. When $a \to 0$, explain the behavior of the posterior mean.

5. (10%) Let $X_1, \dots, X_n$ be iid $N(0,1)$, $n \geq 1$. Let

$$U_n = \frac{\sqrt{n}(X_1 + \cdots + X_n)}{X_1^2 + \cdots + X_n^2}.$$

Show that U_n converges to the standard normal distribution as n goes to infinity.

6. (10%) Let $X_1, \dots, X_n$ be iid Bernoulli(p) with $0 < p < 1$. Let $U_n = \frac{\sum_{i=1}^{n} X_i}{n}$. Find the limiting distribution of $\frac{1}{\sqrt{U_n}}$.

7. Suppose $Y = X\beta + \varepsilon$ where Y is a random vector of length n, X is an $n \times p$ matrix with rank p, β is a vector of length p, and ε is an $n \times 1$ random vector with independently normally distributed components $N(0, \sigma^2)$.

a) (5%) Derive the maximum likelihood estimators of β and σ^2, respectively.

b) (5%) Let $\hat{\beta}$ be your estimator of β from part (a). Derive its sampling distribution.

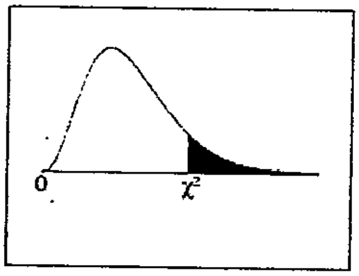
c) (5%) Let $\hat{\sigma}^2$ be your estimator of σ^2 from part (a). Derive the sampling distribution of $\frac{n\hat{\sigma}^2}{\sigma^2}$.

d) (5%) Let $\hat{Y} = X\hat{\beta}$ and $R = Y - \hat{Y}$. Show that $\hat{Y}$ and R are independent.

e) (5%) Are $\hat{\beta}$ and $\hat{\sigma}^2$ independent? Why or why not?

Probability of χ^2 distribution table:

Chi-Square Distribution Table



The shaded area is equal to α for $\chi^2 = \chi^2_\alpha$.

df	$\chi^2_{.995}$	$\chi^2_{.990}$	$\chi^2_{.975}$	$\chi^2_{.950}$	$\chi^2_{.900}$	$\chi^2_{.100}$	$\chi^2_{.050}$	$\chi^2_{.025}$	$\chi^2_{.010}$	$\chi^2_{.005}$
1	0.000	0.000	0.001	0.004	0.016	2.706	3.841	5.024	6.635	7.879
2	0.010	0.020	0.051	0.103	0.211	4.605	5.991	7.378	9.210	10.597
3	0.072	0.115	0.216	0.352	0.584	6.251	7.815	9.348	11.345	12.838
4	0.207	0.297	0.484	0.711	1.064	7.779	9.488	11.143	13.277	14.860
5	0.412	0.554	0.831	1.145	1.610	9.236	11.070	12.833	15.086	16.750