編號： 251

國立成功大學 107 學年度碩士班招生考試試題

系　　所：統計學系

考試科目：數學

考試日期：0206，節次：1

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※ 考生請注意：本試題不可使用計算機。　請於答案卷(卡)作答，於本試題紙上作答者，不予計分。

Please Show ALL of your works.

1. (10%) Let n be a positive real number such that $n > 1$. Please show that
$$(1+x)^n > 1 + nx, \forall x \in (0, \infty).$$

2. (10%) Let f be the function defined as follow:
$$f(x) = \frac{\int_{x}^{x^2} e^{t^2}\,dt}{x-1}, \forall x \neq 1$$
Find a value of $f(1)$ such that the function $f(x)$ is continuous at $x = 1$.

3. (10%) Evaluate the following limits.

(a) $\lim_{x\to\infty}\left(\frac{x^2-4}{x^2-1}\right)^{x^2+1}$　　(b) $\lim_{x\to 1}\frac{1}{x-1}\left(\int_{\frac{1}{2}}^{x}\frac{1}{\sqrt{2t-t^2}}\,dt - \frac{\pi}{6}\right)$

4. (20%) Evaluate the following integrals.

(a) $\int_0^{\pi/4}\frac{1}{(\tan^2\theta+4\tan\theta+3)\cos^2\theta}\,d\theta$　　(b) $\int_{\frac{1}{2}}^{1}\frac{1}{\sqrt{2x-x^2}}\,dx$

(c) $\int \cot^2 x\csc^6 x\,dx$　　(d) $\int \tan^{-1} x\,dx$

5. (10%) A solid is generated by rotating about the x-axis the region under the curve $y = f(x)$, where f is a positive function and $x \geq 0$. The volume generated by the part of the curve from $x = 0$ to $x = c$ is c^2 for all $c > 0$. Find the function f.

6. We are given the LU decomposition of a matrix B as
$$B = LU = \begin{bmatrix}1&0&0&0\\2&1&0&0\\2&1&1&0\\3&2&4&1\end{bmatrix}\begin{bmatrix}1&2&0&1&2&1\\0&0&2&2&0&0\\0&0&0&0&0&1\\0&0&0&0&0&0\end{bmatrix}$$

(a) What is the rank of matrix B? Show your work. (3%)

(b) Find a basis for the null space of B. Show your work. (7%)

(c) Find a basis for the column space of B. Show your work. (5%)

7. Two $n \times n$ real matrices A and B are called simultaneously diagonalizable if there an invertible matrix $D \in \mathbb{R}^{n\times n}$ such that $D^{-1}AD$ and $D^{-1}BD$ both are diagonal matrices. Let A and B be two $n \times n$ real matrices. Please show:

(a) If A and B are simultaneously diagonalizable, then $BA = AB$. (4%)

(b) If $BA = AB$ and if A has n different eigenvalues, then A and B are simultaneously diagonalizable. (6%)

8. (15%) Let Σ be a real symmetric $n \times n$ matrix. Show that the following three statements are equivalent.

(a) All the eigenvalues of Σ are positive.

(b) For every nonzero $x \in \mathbb{R}^n$, one has $x^T\Sigma x > 0$.

(c) There exists an invertible matrix D such that $\Sigma = DD^{\mathrm{T}}$.