【不可使用計算機】*作答前請先核對試題、答案卷(試卷)與准考證之所組別與考科是否相符！！

1. (20 points) Let $f(x,y,z)=6x^2-4xy+7y^2+4xz+5z^2$.

a) (5%) Find a real symmetric matrix A such that $f(x,y,z)=(x\,y\,z)A\begin{pmatrix}x\\y\\z\end{pmatrix}$.

b) (5%) Determine if A is positive definite or not.

c) (5%) Let 1 and 2 be eigenvalues of A corresponding to the eigenvectors $\begin{pmatrix}2\\1\\-2\end{pmatrix}$ and $\begin{pmatrix}1\\2\\2\end{pmatrix}$ respectively. Find the third eigenvalue of A and a corresponding eigenvector.

d) (5%) Given a positive real number a, describe the level surface $f(x,y,z)=a$.

2. (15 points) Let $A=\begin{pmatrix}5&-2&6&11&-17\\0&0&5&6&-7\\2&-3&10&12&-17\\1&-1&2&5&-6\\2&-2&5&8&-11\end{pmatrix}$ with the characteristic polynomial $f_A(x)=(x-1)(x-2)^4$

a) (5%) Find the possible Jordon forms of 5×5 matrices with the same characteristic polynomial as A.

b) (5%) Using the following information to determine the Jordon form of A.

$$\ker(A-2I)=\text{span}\left\{\begin{pmatrix}3\\-1\\1\\0\\1\end{pmatrix},\begin{pmatrix}-1\\-2\\-2\\1\\0\end{pmatrix}\right\},(A-2I)^2=\begin{pmatrix}-2&3&-7&-10&16\\2&-3&7&10&-16\\0&0&0&0&0\\-2&3&-7&-10&16\\-2&3&-7&-10&16\end{pmatrix}$$

Here I is the identity matrix.

c) (5%) Find the minimal polynomial of A.

3. (15 points) Consider the following complex matrix

$$A=\begin{pmatrix}0&0&0\\1&0&-a\\0&1&a+1\end{pmatrix}$$

a) (5%) Find the eigenvalues of A.

b) (10%) Determine for which a, A is diagonalizable. (For each case, you have to explain why A is diagonalizable or not diagonalizable.)

4. (15 points) Suppose we know 20604, 53227, 25755, 20927 and 78421 are divisible by 17. Show that the determinant $\begin{vmatrix}2&5&2&2&7\\0&3&5&0&8\\6&2&7&9&4\\0&2&5&2&2\\4&7&5&7&1\end{vmatrix}$ is also divisible by 17.

5. (15 points) Let $A=\begin{pmatrix}1&2&3&4\\1&2&1&3\\1&3&1&1\end{pmatrix}$ be the matrix representing a linear transformation from $\mathbb{R}^4$ to $\mathbb{R}^3$ corresponding to the standard bases $\left\{\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right\}$ and $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$ of $\mathbb{R}^4$ and $\mathbb{R}^3$, respectively. Let $\mathcal{S}=\left\{\begin{pmatrix}1\\1\\0\\0\end{pmatrix},\begin{pmatrix}1\\0\\1\\0\end{pmatrix},\begin{pmatrix}0\\1\\1\\0\end{pmatrix},\begin{pmatrix}1\\0\\0\\1\end{pmatrix}\right\}$ and $\mathcal{T}=\left\{\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}2\\1\\1\end{pmatrix},\begin{pmatrix}1\\2\\1\end{pmatrix}\right\}$ be another bases of $\mathbb{R}^4$ and $\mathbb{R}^3$, respectively. Find the matrix representation of A corresponding to the bases $\mathcal{S}$ and $\mathcal{T}$.

6. (20 points) Let A be a $n\times n$ skew-symmetric real matrix.

a) (10%) Prove that if n is odd, then $\det(A)=0$.

b) (10 %) Is the same assertion true if n is even. Justify your answer!