國立政治大學 107 學年度 碩士班招生考試試題

考試科目	線性代數	系別	應用數學系	考試時間	2月3日（星期六）第二節

注意事項：

- 作答時，請於答案卷上標明題號，並請勿任意更改題目符號，且請詳列過程，只有答案不給分。請盡量清楚完整回答你會的問題，不要只是每題回答一小部份。
- 本試題共有 5 個問題, 總計 100 分。

1. We say that a linear operator T on V is a **projection**, if there are two subspaces W_1, W_2 of V and $V = W_1 \oplus W_2$, such that for all $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$ with $\mathbf{x}_1 \in W_1$, $\mathbf{x}_2 \in W_2$, we have $T(\mathbf{x}) = \mathbf{x}_1$. In this case, we called that T is the **projection on** W_1 **along** W_2.
   (a) (10 %) Let $T\colon V \to V$ be a projection. Show that $T^2 = T$.
   (b) (10 %) Let W be a subspace of a finite dimensional vector space of V. Show that $V = W \oplus W^{\perp}$. Define the projection T on W along $W^{\perp}$.
   (c) (10 %) Let W_1, W_2 be subspaces of $\mathbb{R}^3$, where $W_1 = \text{span}\{(1,0,0),(1,0,1)\}$ and $W_2 = \text{span}\{(1,1,1)\}$. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the projection on W_1 along W_2. Let β be the standard ordered basis for $\mathbb{R}^3$. Find the matrix representation $[T]_\beta$ for T.

2. Let V be a vector space over $\mathbb{F}$. Let

$$V^* = \{f\colon V \to \mathbb{F} \mid f \text{ is a linear transformation.}\}$$

Note that V^* is also a vector space. For a subset $S \subset V$, the **annihilator** S° of S is defined by

$$S^\circ = \{f \in V^* \mid f(\mathbf{x}) = \mathbf{o} \text{ for all } \mathbf{x} \in S\}.$$

   (a) (10 %) Show that S° is a subspace of V^*.
   (b) (10 %) Let W be a subspace of a finite dimensional vector space of V. Show that

$$\dim W + \dim W^\circ = \dim V.$$

3. (10 %) Let $T\colon V \to V$ be a linear operator on a finite dimensional vector space V. Let β, γ be two bases of V. Show that $\det([T]_\beta) = \det([T]_\gamma)$.

4. (20 %) Let A be a real $n \times n$ matrix. Show that A is invertible if and only if 0 is an eigenvalue of A.

5. (20 %) A real $n \times n$ matrix A is called **positive definite** if $\mathbf{x}^T A\mathbf{x} > 0$ for each nonzero vector $\mathbf{x} \in \mathbb{R}^n$. Let A be an $n \times n$ positive definite matrix. For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, define $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{y}^T A\mathbf{x}$. Show that $\langle \cdot, \cdot \rangle$ is an inner product on $\mathbb{R}^n$.

備註	一、作答於試題上者，不予計分。 二、試題請隨卷繳交。