國立臺灣師範大學 100 學年度碩士班招生考試試題

科目：**高等微積分**　　　　　　適用系所：**數學系**

注意：1.本試題共 2 頁，請依序在答案卷上作答，並標明題號，不必抄題。2.答案必須寫在指定作答區內，否則不予計分。

1. (15 分) Consider a bounded closed rectangle $E \subseteq \mathbb{R}^2$ and suppose $f : E \to \mathbb{R}$ is continuous. Prove that f attains its maximum and minimum, f has the intermediate value property, and f is uniformly continuous on E.

2. (10分)

   (a) Let $\{s_n\}$ be a real sequence with

   $$s_1 \geq s_2 \geq \cdots \geq s_n \geq s_{n+1} \geq \cdots \geq 0.$$

   Prove that if $\sum_{n=1}^{\infty} s_n$ converges, then $\lim_{n\to\infty} ns_n = 0$, but the converse statement does not hold.

   (b) Determine whether the following series converges or diverges:

   $$\sum_{n=1}^{\infty} \frac{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}}{n}.$$

3. (10 分) Suppose $f : (a,b) \to (c,d)$ is differentiable, $f'(x) \neq 0$ for each $x \in (a,b)$, and f is onto. Prove that f is a homeomorphism and the inverse function $f^{-1} : (c,d) \to (a,b)$ is differentiable with the derivative

   $$\left(f^{-1}\right)'(y) = \frac{1}{f'(x)}, \qquad \text{where } y = f(x).$$

4. (15 分)

   (a) Prove that if $\{K_n\}$ is a sequenece of nonempty compact subsets of a metric space such that $K_1 \supset K_2 \supset \cdots$, then $\cap_{n=1}^{\infty} K_n \neq \emptyset$.

   (b) Suppose that K is a compact metric space, and

      (i) $\{f_n\}$ is a sequence of real-values continuous functions on K,

      (ii) $\{f_n\}$ converges pointwise to a continuous function f on K, and

      (iii) $f_n(x) \geq f_{n+1}(x)$ for all $x \in K$, $n = 1, 2, \ldots$.

   Prove that $f_n \to f$ uniformly on K.

(背面尚有試題)

5. (15 分) Let A and B be two nonempty subsets of negative real numbers, with $\inf A = -3$ and $\inf B = -5$. Define another set $C = \{a \times b \mid a \in A, b \in B\}$. Which of the following statements:

(a) $\inf C = 15$ (b) $\sup C = 15$ (c) $\inf C = 0$ (d) $\sup C = 0$

is true? Indicate the correct statement and prove it.

6. (10 分) Let $g(x) = \mathrm{e}^{-2x} \cdot \sin\left(\frac{x}{2}\right)$. Show that $g(x) < \frac{1}{\sqrt{17}}$ for all $x \geq 0$.

7. (10 分) Let h be a real-valued function whose second derivative h'' is continuous over $\mathbb{R}$. Given that $h(0) = 3$, $h'(0) = -4$, $h(2) = -2$ and $h'(2) = 5$, evaluate the integral:

$$\int_0^2 x h''(x)\,\mathrm{d}x.$$

8. (15 分) Let B be the solid ball $\{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \leq r^2\}$ of radius $r > 0$ in $\mathbb{R}^3$. Evaluate the triple integral:

$$\iiint_B (x^2+y^2+z^2)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

(試題結束)