國立臺灣師範大學 100 學年度碩士班招生考試試題

科目：機率與統計　　　　　　　　　　　適用系所：數學系

注意：1.本試題共 1 頁，請依序在答案卷上作答，並標明題號，不必抄題。2.答案必須寫在指定作答區內，否則不予計分。

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1. (15分) Let Y denote a uniform random variable on (0,1). Suppose that the conditional distribution of X, given $Y = p$, is a binomial with parameters n and p.

   (a) Find the expectation and variance of X.

   (b) Find the distribution of X using moment generating function.

2. (20分) Let X and Y be independent exponential random variables with respective means $1/\lambda_1$ and $1/\lambda_2$.

   (a) Compute the distribution of $Z = min(X, Y)$.

   (b) Find the conditional distribution of Z given that $Z = X$.

3. (10分) A has 1 coin and B has 4 coins. A and B bet on the outcomes of successive flips of a fair coin. After each flip, the winner collects one coin from the loser. They continue to do this until one of them runs out of coins. Find the probability that A ends up with all the coins.

4. (15分) Let X be a random variable with the probability function

$$f(x|\theta) = \left(\frac{\theta}{2}\right)^{|x|} (1-\theta)^{1-|x|}, x = -1, 0, 1, 0 \le \theta \le 1.$$

   (a) Is X a complete sufficient statistic? Explain your reasons.

   (b) Is $|X|$ a complete sufficient statistic? Explain your reasons.

   (c) Examine whether $f(x|\theta)$ belongs to the exponential family (or exponential class).

5. (20分) Let $\bar{X}$, $\bar{Y}$, S_X^2, S_Y^2 be the respective sample means and unbiased sample variances using independent samples of size n and m from the normal distributions $N(\mu_X, \sigma^2)$, and $N(\mu_Y, \sigma^2)$, where μ_X, μ_Y, and σ^2 are unknown.

   (a) Based on the above sample statistics, derive an unbiased estimator of σ^2 which has smaller variance than S_X^2 and S_Y^2.

   (b) Construct a 100(1-α)% confidence interval of σ^2 using the point estimator in (a).

6. (20分) Let $Y_1 < Y_2 < \cdots < Y_n$ be the order statistics of a random sample from a uniform distribution on $(0, \theta]$, where $\theta > 0$.

   (a) Derive the likelihood ratio test statistic Λ for testing $H_0 : \theta = \theta_0$ against $H_1 : \theta \ne \theta_0$.

   (b) When H_0 is true, give the distribution of $-2 \ln \Lambda$.