國立臺灣師範大學 100 學年度碩士班招生考試試題

科目：**應用數學**　　　　　　　　適用系所：**物理學系**

注意：1.本試題共 2 頁，請依序在答案卷上作答，並標明題號，不必抄題。2.答案必須寫在指定作答區內，否則不予計分。

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1. Given a symmetric matrix

$$A = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix},$$

(a) Find the eigenvalues and the corresponding eigenvectors of the symmetric matrix A. Remember to normalize each eigenvector to a unit vector. (10 marks)

(b) Find a matrix U which is resulted from the eigenvectors of A so that UAU^{-1} is a diagonal matrix with the diagonal matrix elements being the eigenvalues of A. Please show your work of calculating UAU^{-1} explicitly. (10 marks)

2. (a) What is the Stokes's theorem in vector analysis? (5 marks)

(b) Verify the Stokes's theorem for $\vec{A} = 2y\vec{\imath} + 3x\vec{\jmath} - z^2\vec{k}$ by considering the surface S of the upper half sphere $x^2 + y^2 + z^2 = 9$ and its boundary C. The normal vectors for the surface S are chosen to point outward. (15 marks)

3. Consider the following second-order ordinary differential equation (2nd-order ODE)

$$y''(x) + y(x) = 0.$$

(a) Find a general solution for this 2nd-order ODE without using the method of power series. (5 marks)

(b) Using the method of power series to solve this 2nd-order ODE. Verify that the solutions obtained by these 2 different methods are actually identical. (15 marks)

（下一頁尚有題目，請翻頁後繼續作答）

4. Evaluate the following two integrals (each counts 10 marks)

$$(a)\quad \int_0^{2\pi} \frac{1}{5-4\cos\theta}\, d\theta,$$

$$(b)\quad \int_0^{\infty} \frac{1}{1+x^4}\, dx.$$

Hint: you might want to use the residue theorem in complex analysis. You will receive the full credit as long as the method you use to calculate these integrals leads to correct answers.

5. Given a function $f(t)$ defined for all $t \geq 0$, its Laplace transform $\mathcal{L}(f(t))(s)$, which is conventionally denoted by $F(s)$, is defined by

$$F(s) = \mathcal{L}(f(t))(s) = \int_0^{\infty} e^{-st} f(t)dt, \qquad s > 0.$$

(a) Show that

$$\mathcal{L}(f'(t))(s) = sF(s) - f(0),$$
$$\mathcal{L}(f''(t))(s) = s^2F(s) - sf(0) - f'(0),$$

where $f'(t)$ and $f''(t)$ stand for the first derivative and the second derivative of $f(t)$ (with respect to t), respectively. You might assume that the Laplace transforms of $f(t), f'(t)$, and $f''(t)$ exist. (10 marks)

(b) Use Laplace transform to solve the following second-order ordinary differential equation (2nd-order ODE) with the given initial conditions. (10 marks)

$$f''(t) + f(t) = t, f(0) = 0, f'(0) = 2.$$

Notice if you solve (b) by methods other than Laplace transform, then the most credit you can get is 4 marks (out of 10 marks).