Part I：填空題（每格5分，共50分）

注意事項:

(1) 此部分不須計算過程。

(2) 請不要使用「選擇題作答區」作答。

(3) 請自行於作答區第一頁「選擇題作答區」的下面製作如下的填空題作答區：

(a)	(b)	(c)	(d)	(e)
(f)	(g)	(h)	(i)	(j)

1. Let X and Y be two random variables. Suppose that

$$\mathrm{E}(X|Y) = 1.5 + 0.5Y,\ \mathrm{Var}(X|Y) = 0.75Y^2,\ \mathrm{E}(Y) = 0, \text{ and } \mathrm{E}(Y^2) = 1.$$

Then we can obtain that $\mathrm{E}(X) =$ ___(a)___, $\mathrm{Var}(X) =$ ___(b)___, and $\mathrm{Cov}(X,Y) =$ ___(c)___. Now suppose that $\mathrm{E}(Y|X) = \alpha + \beta X$, where α and β are two non-stochastic parameters. Then $\alpha =$ ___(d)___ and $\beta =$ ___(e)___.

2. Let $X_i \sim \mathrm{i.i.d.}\, N(0,1)$, $i = 1,\ldots,n$. Also let $f(x_1,\ldots,x_n)$ be the corresponding joint probability density function. Then $f(x_1,\ldots,x_n) =$ ___(f)___. Consider the sample mean: $\bar{X}_n = n^{-1}\sum_{i=1}^{n} X_i$. We can find that $\mathrm{E}(\bar{X}_n) =$ ___(g)___ and $\mathrm{E}[\sum_{i=1}^{n}(X_i - \bar{X}_n)^2] =$ ___(h)___. Now suppose that $M(t)$ is the moment generating function of $\bar{X}_n$. Then $M(t) =$ ___(i)___. Let $Y_1 < Y_2 < \cdots < Y_n$ be the order statistics of $X_1,\ldots,X_n$. Suppose that $g(y_n)$ is the probability density function of Y_n. Then $g(y_n) =$ ___(j)___.

Part II：計算問答說明題（50分）

Note: You should carefully state the reasons or calculations in the following questions in order to get the points. A short answer, such as "Yes" or "No" will NOT receive any point.

1. Consider the simple linear regression model $Y_i = \beta_0 + \beta_1 X_i + u_i$, $i = 1, \ldots, n$. Assume all the general assumptions hold for the linear regression. Let

$$Y = \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix},\ X = \begin{pmatrix} 1 & X_1 \\ \vdots & \vdots \\ 1 & X_n \end{pmatrix},\ u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \text{ and } \beta = \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix},$$

then the matrix form of the model is $Y = X\beta + u$, where $\mathrm{Var}(u) = \sigma^2 I_n$. The ordinary least squares (OLS) estimator of β is

$$\hat{\beta} = \begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{pmatrix} = (X^{\mathrm{T}}X)^{-1}X^{\mathrm{T}}Y.$$

(a) Find $(X^{\mathrm{T}}X)^{-1}$ and $X^{\mathrm{T}}Y$. (10%)

(b) Use the result from (a) to show that $\hat{\beta}_0 = \overline{Y} - \hat{\beta}_1\overline{X}$. (10%)

(c) Given the matrices $\hat{\beta}$, Y and $X^{\mathrm{T}}Y$, how can we use them to compute the coefficient of determination R^2? (10%)

(d) Show that $\mathrm{Var}(\hat{\beta}) = \sigma^2(X^{\mathrm{T}}X)^{-1}$. (10%)

(e) If we want to test the hypothesis $H_0: \beta_0 + \beta_1 = 1$ against the alternative hypothesis $H_1: \beta_0 + \beta_1 \neq 1$. How can we use the above result to conduct the test? Please be specific about the test statistics. (10%)