Total points: 100

1. 1. Given a surface S in $\mathbb{R}^3$ by the equation $z = f(x,y)$. We write $f_x = p$, $f_y = q$ and $\sqrt{1+p^2+q^2} = M$ for convenience. a) (20 points) Find the first fundamental form (of S) and the field $\mathbf{N}$ of unit vectors normal to S (up to $\pm$ sign). b) (10 points) Find the second fundamental form. Express your answers in terms of p, q, M and their derivatives.

2. Let $\mathbf{N}(u,v)$ be the field of unit vectors normal to a surface S of $\mathbb{R}^3$ (in terms of local parameters u, v). a) (15 points) Show that S is of Gaussian curvature zero if and only if the triple product $(\mathbf{N}\mathbf{N}_u\mathbf{N}_v)$ (i.e. the determinant of the matrix formed by the three vectors) is identically zero on S. b) (10 points) Is the surface $z = ax + by + D(ux+vy)^m + E(ux+vy)^n$ where $a, b, u, v, D, E \in \mathbb{R}$ and $m, n \in \mathbb{N}$, of Gaussian curvature zero?

3. (25 points) Let $\gamma = \gamma(s)$ be a space curve parametrized by arc length s with the Frenet trihedron $T(s), N(s), B(s)$ (i.e. tangents, principal normal and binormal). Suppose S is a surface (in $\mathbb{R}^3$) containing γ. With respect to S, we write $\frac{DT}{ds}(s)$ for the covariant derivative of T (at s). Show that $\frac{DT}{ds}(s) \equiv 0$ if and only if the tangent plane $T_{\gamma(s)}S$ is the same as the plane spanned by $T(s)$ and $B(s)$.

4. (20 points) Let $\gamma(t)$, $0 \le t \le 1$, be a simple closed curve (i.e. $\gamma(0) = \gamma(1)$ without self-intersection) bounding a region Ω on a surface S (in $\mathbb{R}^3$). Assume that Ω is contained in a coordinate chart of S and that, with $w(0) \in T_{\gamma(0)}S$, $w(t)$ is the parallel transport of $w(0)$ along γ. Write θ for the angle from $w(0)$ to $w(1)$. Show that $\theta = \int\int_\Omega K d\sigma$ (up to $n \cdot 2\pi$, $n \in \mathbb{Z}$) where K is the Gaussian curvature of S and $d\sigma$ is the surface area element.

試題隨卷繳回