一、Linear Algebra (50%)

1. (10%) Let $A=[\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3\}$ be a 5×3 matrix. If

$$\mathbf{b}=\mathbf{a}_1+\mathbf{a}_2=\mathbf{a}_2+\mathbf{a}_3$$

then what can you conclude about the number of solutions of the linear system $A\mathbf{x}=\mathbf{b}$? Explain.

2. (5%) Derive the line in R^3 that contains the point $P(-1,6,0)$ and is orthogonal to the plane $4x-z=5$.

3. (10%) Are there values of r and s for which

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & r-2 & 2 \\ 0 & s-1 & r+2 \\ 0 & 0 & 3 \end{bmatrix}$$

has rank 1? Has rank 2? If so, find those values.

4. There is a set $M=\left\{\begin{bmatrix} m_1 & m_2 \\ m_3 & m_4 \end{bmatrix}\middle| m_i\in\{0,1\}, i=1,2,3,\text{and } 4\right\}$,
   a. (10%) In M, find all matrices with two distinct eigenvalues 0 and 1.
   b. (5%) In M, find all diagonalizable matrices with only one eigenvalue 0 (algebraic multiplicity = 2).
   c. (10%) Constructing an LU-decomposition for a 2×2 matrix in M with $\prod_{i=1}^{4} m_i = 1$.

二、微分方程

1. (10%) Answer the following questions:

   (a) (5%) Find the general solution y_c for

   $$xy'' - 4y' = 0$$

   (b) (5%) Find a particular solution y_p for

   $$xy'' - 4y' = x^4$$

   using the method of **variation of parameters**.

2. (5%) Find $f(t)$ by solving the following integral equation using the Laplace transform.

   $$f(t) = 3t^2 - e^{-t} - \int_0^t f(\tau)e^{t-\tau}d\tau.$$

3. (5%) Use Laplace transform to compute $e^{\mathbf{A}t}$ for

   $$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}$$

4. (5%) Solve the general solution for the following differential equation

   $$\frac{d^4y}{dx^4} + 2\frac{d^2y}{dx^2} + y = 0. \tag{1}$$

5. (10%) Find the Fourier integral representation of the function

   $$f(x) = \begin{cases} 0, & x < -1 \\ -1, & -1 < x < 0 \\ 2, & 0 < x < 1 \\ 0, & x > 1 \end{cases}$$

6. (15%) Find the eigenvalues and eigenfunctions of the boundary-value problem

   $$y'' + y' + \lambda y = 0, \quad y(0) = 0, \quad y(2) = 0.$$