國立中山大學107學年度碩士暨碩士專班招生考試試題

科目名稱：線性代數【通訊所碩士班甲組】　　　　題號：437006

※本科目依簡章規定「可以」使用計算機（廠牌、功能不拘）（混合題）　　　　共2頁第1頁

In the following, boldface capital and lower-case letters denote matrices and vectors, respectively.
For questions 1~3, please select the best answer from the choices provided. (單選)
For questions 4~13, please provide both answers and justifications.

1. (5%) Suppose a 4×5 matrix **A** has rank 4. Then the equation **Ax**=**b**
   (a) always has a unique solution.
   (b) always has no solution.
   (c) always has many solutions.
   (d) sometimes but not always has a unique solution.
   (e) sometimes but not always has many solutions.

2. (5%) Suppose a 3×5 matrix **A** has rank 3.
   (a) The orthogonal complement of the range space of **A** is a 3-dimensional space.
   (b) The null space of **A** is a 3-dimensional space.
   (c) The column space of **A** is a 3-dimensional space.
   (d) The kernel of **A** is a 3-dimensional space.
   (e) The orthogonal complement of the kernel of **A** has dimension 2.

3. (5%) Which of the following matrices is a linear combination of $\begin{bmatrix} 3 & -1 \\ 5 & 2 \end{bmatrix}$, $\begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix}$, and $\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$?
   (a) $\begin{bmatrix} 2 & 3 \\ -4 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ (d) $\begin{bmatrix} -4 & 6 \\ -13 & 4 \end{bmatrix}$ (e) $\begin{bmatrix} 3 & -1 \\ 8 & 2 \end{bmatrix}$.

4. (10%) Let **A** and **B** be 3×3 matrices with det(**A**) = 5, det(**B**) = 10, and det(**A**+**B**) = 60. Decide the following values.
   (a) (5%) det(**A**+**A**).
   (b) (5%) $\det(\mathbf{A}^2\mathbf{B}+\mathbf{A}\mathbf{B}^2)$.

5. (10%) Let **A** be an 2x2 real symmetric matrix

$$\mathbf{A} = \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}.$$

   (a) (5%) Find $\mathbf{A}^{\frac{1}{2}}$.
   (b) (5%) $F(\mathbf{x}) = \frac{\mathbf{x}^T\mathbf{A}\mathbf{x}}{\mathbf{x}^T\mathbf{x}}$, find the maximum and minimum values of $F(\mathbf{x})$ over the set of nonzero vectors in R^2.

6. (10%) Let dim(Z) denote the dimension of the vector space Z and rank(**C**) denote the rank of the matrix **C**. Show that:
   (a) (5%) If X and Y are subspaces of a vector space V, then $\dim(X+Y)=\dim(X)+\dim(Y)-\dim(X\cap Y)$.
   (b) (5%) rank(**A**+**B**) ≤ rank(**A**)+ rank(**B**) where **A** and **B** are $m \times n$ matrices. (Hint: use the result in (a))

7. (5%) Given the following matrix:

$$\begin{bmatrix} 2 & 1-i \\ 1+i & 1 \end{bmatrix}.$$

Determine whether it is Hermitian, unitary, singular and positive definite.
Please explain your reasons to each answer.

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8. (5%) Let $\mathbf{u}_1 = (1,1,1)^T$, $\mathbf{u}_2 = (1,2,2)^T$, $\mathbf{u}_3 = (2,3,4)^T$.
   (a) (2%) Find the transition matrix corresponding to the change of basis from $[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]$ to $[\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$.
   (b) (3%) Find the coordinates of $(2,3,2)^T$ with respect to $[\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$.

9. (10%) Suppose that $\mathbf{A}$ is a 5×3 real matrix of rank 3.Let $\mathbf{W} = \mathbf{A}^T\mathbf{A}$ and $\mathbf{S} = \mathbf{A}\mathbf{A}^T$.
   (a) (3%) Find the ranks of $\mathbf{W}$ and $\mathbf{S}$.
   (b) (3%) Explain why $\lambda = 0$ is an eigenvalue of $\mathbf{S}$.
   (c) (4%) What is the (algebraic) multiplicity of the eigenvalue $\lambda = 0$ of $\mathbf{S}$?

10. (5%) Find the Jordan canonical form of

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

11. (10%) Let $\mathbf{M}$ be the vector space of all (3×3) real-valued matrices over the real field. Let $\mathbf{T}: \mathbf{M} \to \mathbf{M}$ be a linear transformation given by

$$\mathbf{T}(\mathbf{X}) = \mathbf{A}\mathbf{X}, \text{ where } \mathbf{A} = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}.$$

   (a) (5%) Find a basis for the kernel of $\mathbf{T}$.
   (b) (5%) For each eigenvalue of $\mathbf{T}$, find a basis for the corresponding eigenspace.

12. (10%) If $\mathbf{K} = \begin{bmatrix} 1 & 2 & 2^2 & 2^3 & 2^4 & 2^5 \\ 2 & 1 & 2 & 2^2 & 2^3 & 2^4 \\ 2^2 & 2 & 1 & 2 & 2^2 & 2^3 \\ 2^3 & 2^2 & 2 & 1 & 2 & 2^2 \\ 2^4 & 2^3 & 2^2 & 2 & 1 & 2 \\ 2^5 & 2^4 & 2^3 & 2^2 & 2 & 1 \end{bmatrix}$, find $\det(\mathbf{K})$.

13. (10%) On $P_2(R)$, consider the inner product given by $<p, q> = \int_0^1 p(x)q(x)dx$.
   (a) (5%) Show that the basis $(1, x, x^2)$ is NOT orthonormal.
   (b) (5%) Apply the Gram-Schmidt procedure to $(1, x, x^2)$ to produce an orthonormal basis of $P_2(R)$.
   Note that $P_2(R)$ is the set of all polynomials of degree 2 with real valued coefficients.