國立中山大學107學年度碩士暨碩士專班招生考試試題

科目名稱：工程數學乙【電機系碩士班乙組】　題號：431001

※本科目依簡章規定「可以」使用計算機（廠牌、功能不拘）（問答申論題）　共2頁第1頁

第1至4題請照下列指示作答

- 所有對定義或問答題的做答，都要用該題目所給定的、和線性代數課程中所學的數學符號(包含∃和∀的使用)，完全正確回答才給分。
- 對於集合的數學描述要完整且完全正確才給分。例如用集合 $\{s\in\mathbb{C}\,|\,s=\sigma+j\omega,\ \sigma>0\ and\ \omega\in\mathbb{R}\}$ 表示複數平面的開右半平面。
- 所有需要推導或證明的子題都已用**粗體**標示出來，其他子題都只需用數學符號(而非用文字敘述)明確且精簡地做答，否則不予計分。
- 占分"≤ 2%"的證明題要完全正確才給分。占分"≥ 3%"的證明題才會給部分成績。

1. (12%) Let $A\in\mathbb{R}^{m\times n}$ and $\mathbf{b}\in\mathbb{R}^m$.

(a) (2%) (i) What is the mathematic relationship of dimensions of two subspaces $R(A)$ and $N(A)$ for any A matrix considered here? (ii) What is the mathematic relationship described in the Fundamental Subspaces Theorem about matrix A?

(b) (2%) Use the Fundamental Subspaces Theorem to **show** that $N(A^TA)\subset N(A)$.

(c) (2%) (i) What is the condition on $R(A)$ or $N(A)$ that is equivalent to the existence of solution to $A\mathbf{x}=\mathbf{b}$? (ii) What is the condition on $R(A)$ or $N(A)$ that is equivalent to the uniqueness of solution to $A\mathbf{x}=\mathbf{b}$, if it is solvable?

(d) (2+4%) When the equation $A\mathbf{x}=\mathbf{b}$ is unsolvable, we may consider the so-called least squares problem to find a set of solutions, having the least squares error, from solving a normal equation. (i) Use the property and condition mentioned in (b)-(c) to explain why the normal equation is always solvable. (ii) Suppose that $rank(A)=k<\min(m,n)$ and let $A=BC$ be a full rank decomposition of A. Use the known matrices B, C, and $\mathbf{b}$ to describe the unique projection vector $\mathbf{p}$ of $\mathbf{b}$ onto $R(A)$ with the least $\|\mathbf{b}-\mathbf{p}\|_2$.

2. (13%) Let V be a vector space.

(a) (1+2%) Let $\boldsymbol{\delta}$ be a unit element of "+", the addition operation of V. (i) Write the equality condition about $\boldsymbol{\delta}$ as shown in the corresponding axiom. (ii) Let γ be another unit element of "+". **Show** that $\boldsymbol{\delta}=\gamma$. (須註明使用到的所有向量空間定義中的公設，否則不計分)

(b) (3%) Let X and Y be two subspaces of V. (i) Write the definition of $X+Y$. (ii) Write the definition of $X\oplus Y$. (iii) What is the mathematical relationship between $\dim(X+Y)$ and $\dim(X\oplus Y)$?

(c) (2%) Let $\langle\cdot,\cdot\rangle$ be an inner product defined on V. **Show** that the function $f(\mathbf{v}):=\sqrt{\langle\mathbf{v},\mathbf{v}\rangle}$ defined $\forall\mathbf{v}\in V$ satisfies the triangular inequality property.

(d) (2+3%) Consider the vector space $\mathbb{R}^{2\times2}$ with the inner product $\langle A,B\rangle:=trace(A^TB)$ and denote $Y:=\{A\in\mathbb{R}^{2\times2}\,|\,A=A^T\}$. (i) Describe $Y^\perp$ as the span of an orthonormal basis. (ii) What is the matrix, denoted by $P_{Y^\perp}$, that represents the orthogonal projecting operation $\Pi_{Y^\perp}:\mathbb{R}^{2\times2}\to Y^\perp$ along with the subspace Y with respect to the ordered basis $E=\{G,H\}$, where G and H are vectors of the standard bases for Y and $Y^\perp$, respectively?

3. (14%) Let L be a mapping from vector space V to vector space W.

(a) (1+2%) (i) Write the definition of superposition principle and (ii) **show** the implication from the combination of additive and homogeneous properties to the superposition principle.

(b) (2%) Write (i) the definition of $Ker(L)$, the kernel of L, and (ii) the definition of $L(V)$, the image of vector space V, respectively.

(c) (1+2%) Suppose L is linear. (i) Use $Ker(L)$ to describe a mathematic condition that is equivalent to L being one-to-one. (ii) Moreover, **show** the implication from the condition on $Ker(L)$ to L being one-to-one.

(d) (2+4%) Let L be linear with A as its matrix representation with respective to bases $E=[\mathbf{v}_1,\cdots,\mathbf{v}_n]$ and $F=[\mathbf{w}_1,\cdots,\mathbf{w}_m]$ for V and W, respectively. (i) Describe a mathematic condition about A that is equivalent to L being onto. (ii) Moreover, **show** the implication from that condition to L being onto.

4. (11%) Let $A\in\mathbb{C}^{n\times n}$.

(a) (2%) Let $\lambda\in\mathbb{C}$ be an eigenvalue of A with corresponding eigenvector $\mathbf{x}$. **Derive** in details the equation for solving λ. (要求推導出公式、而非背寫出公式卻沒有任何數學的說明)

(b) (2+2%) (i) Let μ be an eigenvalue of matrix A. What is the mathematic notation for describing the number of linearly independent eigenvectors associated with μ? (ii) Let $\{\mu_1,\cdots,\mu_k\}$ be the set of all distinct eigenvalues of matrix A. Use the corresponding notation as in (i) to describe the condition for A being a diagonalizable matrix.

(c) (5%) Suppose $A=B+iC$ is a Hermitian matrix and let $\Omega=\begin{bmatrix} B & C \\ -C & B\end{bmatrix}\in\mathbb{R}^{2n\times 2n}$. **Show** that any eigenvalue λ of Ω is also an eigenvalue of A.

以下第5題到第6題中之所有的提問，需要寫出推導過程或詳細說明理由，答案正確但沒有推導過程或說明不正確，將酌扣分數或不給分。

5. (20%) Consider the following set of differential equations

$$\ddot{x}_1(t) = -2x_1(t) + 2x_2(t)$$
$$\ddot{x}_2(t) = 2x_1(t) + 5x_2(t) + u(t)$$

(a) (15%) Let $u(t)\equiv 0$ and the initial conditions be $x_1(0)=x_2(0)=1$, $\dot{x}_1(0)=\dot{x}_2(0)=0$. Find the solutions of the differential equations.

(b) (5%) Let initial conditions be $x_1(0)=x_2(0)=\dot{x}_1(0)=\dot{x}_2(0)=0$, and $u(t)$ be the unit step function. Does the solutions of the differential equations converge to constant values as time approaches infinity? Justify your answers.

6. (30%) Consider the following set of differential equations

$$\dot{x}_1(t) = (1-\sqrt{x_1(t)^2+x_2(t)^2})x_1(t) - x_2(t)$$
$$\dot{x}_2(t) = (1-\sqrt{x_1(t)^2+x_2(t)^2})x_2(t) + x_1(t)$$

(a) (10%) Express the differential equations using the polar coordinate; i.e. express the equations in terms of (r,θ), where $x_1(t)=r(t)\cos(\theta(t))$ and $x_2(t)=r(t)\sin(\theta(t))$.

(b) (10%) Solve the differential equations for any nonzero initial conditions (x_{10},x_{20}).

(c) (10%) Draw the phase plane portraits for the solutions (x_1,x_2) of the following initial conditions: $(x_{10},x_{20})=(0,0.5)$, $(-1,0)$, and $(1,-1)$.

End of Examination