※ 考生請注意：本試題不可使用計算機。 請於答案卷(卡)作答，於本試題紙上作答者，不予計分。

General Notations:
$\mathbb{R}$: The set of real numbers.
$\mathbb{Z}$: The set of integers.
$\mathbb{Q}$: The set of rational numbers.
$[a,b] := \{x \in \mathbb{R} \mid a \leq x \leq b\}$.
$A \backslash B := \{x \in A \mid x \notin B\}$.

1. (10 points) Compute the following double integral:

$$\int_0^{\frac{1}{2}} \int_{2y}^{1} e^{-x^2}\, dxdy.$$

2. Given a metric space (X, d),
   (a) (5 points) State the definition of a *metric* d on X.
   (b) (5 points) Define *open subsets* of X with respect to d.
   (c) (5 points) Define *compact subsets* of X with respect to d.

3. Given a metric space (X, d), determine whether each of the following statements is true or false. Prove the statement if it is true, and provide counterexample if it is false.
   (a) (10 points) Any intersection of open subsets of X is open.
   (b) (10 points) Any intersection of compact subsets of X is compact.

4. Given a sequence $\{a_n\}$ of real numbers,
   (a) (5 points) State the definitions of $\limsup_n a_n$ and $\liminf_n a_n$.
   (b) (5 points) Prove that if a_n converges to a, then

$$\liminf_n a_n = \limsup_n a_n = a.$$

5. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^2 \cos \frac{1}{x} & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$$

   (a) (10 points) Prove that f is differentiable on $\mathbb{R}$.
   (b) (5 points) Prove that f' is not continuous at $x = 0$.

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6. For $x \in \mathbb{R}$, define the function (x) to be the fractional part of x. That is,

$$(x) = x - [x],$$

where $[x]$ is the *greatest* integer $\leq x$.

(a) (5 points) Prove that (x) is continuous on $\mathbb{R}\backslash\mathbb{Z}$.

(b) (5 points) Given $n \in \mathbb{Z}$, what is the set of *dis*continuity for (nx).

(c) (10 points) Prove that the function

$$\sum_{n=1}^{\infty} \frac{(nx)}{n^3}$$

is continuous on irrational numbers $\mathbb{R}\backslash\mathbb{Q}$.

7. (10 points) Given a continuous function f on $[a, b]$, prove that there exists a sequence of polynomials $\{P_n\}$ so that

$$\lim_{n\to\infty} \int_a^b |f - P_n|^2 \, dx = 0.$$

(Hint: You may use some famous theorem)