※ 考生請注意：本試題不可使用計算機。　請於答案卷(卡)作答，於本試題紙上作答者，不予計分。

1. (10%) There are three doors on the set for a game show. Let's call them A, B and C. There is a car behinds one door. If you pick the correct door you win the prize. You can first pick a door, e.g., A. Then the host of the show must open one of the other doors and reveals that there is no car behind it. The host then asks you whether you want to switch your choice to the other closed door or stay with your original choice of door A. Under this game rule, use the decision tree to show the probability that you will win the car under the following conditions:
   (1) You stay with door A
   (2) You switch to the other closed door

2. (20%) Most medical tests occasionally produce incorrect results, called false positives and false negatives. When a test is designed to determine whether a patient has a certain disease, a false positive result indicates that a patient has the disease when the patient does not have it. A false negative result indicates that a patient does not have the disease when the patient does have it. Consider a medical test that screens for a disease found in 5 people in 1000. Suppose that the false positive rate is 3% and the false negative rate is 1%.
   Let A be the event that the person tests positive for the disease, B_1 the event that the person actually has the disease, and B_2 the event that the person does not have the disease.
   (1) Compute $P(A|B_1)$, $P(A'|B_1)$, $P(A|B_2)$, $P(A'|B_2)$, $P(B_1)$, $P(B_2)$. (6%)
   (2) Find the probability that a person who does have the condition will test positive for it. (2%)
   (3) Find the probability that a person who does not have the condition will have a negative test result. (2%)
   (4) Give the Bayes' Theorem. (2%)
   (5) Find the probability of a randomly chosen person who tests positive actually has the disease. (4%)
   (6) Find the probability of a randomly chosen person who tests negative does not indeed have the disease? (4%)

3. (10%) Suppose that $X_1, X_2, \ldots X_n$ are independent random variables having the same distribution function. Let $\mu=E[X_i]$ and $\sigma^2=\mathrm{Var}(X_i)$ denote, respectively, their mean and variance. In statistics, the sample mean is computed as
   $\bar{X} = \sum_{i=1}^{n} \frac{X_i}{n}$. Compute the expected mean square error of the sample mean.

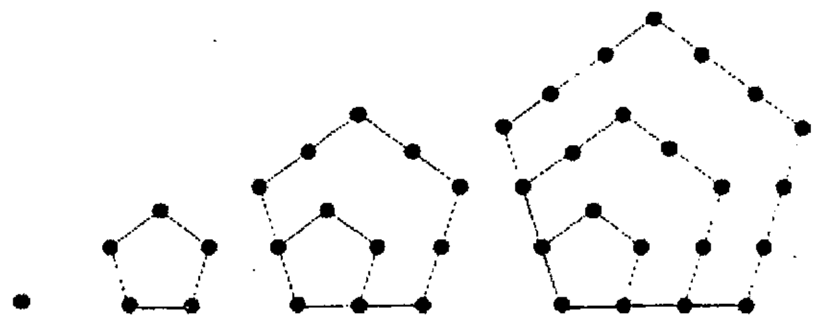
4. (10%) Show that among any six people, there are three mutual friends or three mutual strangers.

5. (10%) Let $X=\{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$
   For $x, y \in X$, set $x R y$ if $x^2<y^2$ or $x=y$.
   (1) Show that R is a partial ordering on X
   (2) Draw a Hasse Diagram of R

6. (10%) Prove that there are infinitely many prime numbers by contradiction

7. (10%) Let p(x), q(x), and r(x) denote the following open statements. For the universe of all integers, determine the truth or falsity of each of the following statements. If a statement is false, give a counterexample.
   p(x): x^2-88x+77=0, q(x): x is odd, r(x): x > 0
   (1) $\forall x\,[q(x) \rightarrow p(x)]$
   (2) $\exists x\,[p(x) \rightarrow q(x)]$
   (3) $\exists x\,[r(x) \rightarrow p(x)]$
   (4) $\forall x\,[\neg q(x) \rightarrow \neg p(x)]$
   (5) $\forall x\,[(p(x) \vee q(x)) \rightarrow r(x)]$

8. (10%) Early members of the Pythagorean Society defined figurate numbers to be the number of dots in certain geometrical configurations. For example, the first four pentagonal numbers of following graph are 1, 5, 12, and 22.



Find a recursive expression for the nth pentagonal number. Guess the closed form for this function and prove your answer to be correct.

9. (10%)For each of the algorithm segments provided, assume that n is a positive integer.
   (1) Compute the actual numbers of additions, subtractions, multiplications, divisions, and comparisons that must be performed when the algorithm segment is executed. For simplicity, however, count only comparisons that occur within if-then statements; ignore those implied by for-next loops.
   (2) Use the theorem on polynomial orders to find an order for the algorithm segment.

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for i := 1 to n do
   for j := 1 to i do
      for k := 1 to j do
         x := i*j*k
      next k
   next j
next i
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