考 試 科 目　基礎數學　系所別　統計學系　考試時間　2 月 2 日（五）第 一 節

1. (12pts). For each of the following statements, determine whether it is true or false. (True or false questions. Do not give explanation)

   (a) Let A and B be both square matrices of the same size. Then rank(AB) = rank(BA).

   (b) Suppose V is a vector space of dimension n and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in V$. Then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is linear independent if and only if $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ spans V.

   (c) Let $f : [0,1] \longrightarrow \mathcal{R}$. If $\lim_{n \to \infty} \Sigma_{k=1}^{n} f(\frac{k}{n})(\frac{1}{n})$ exists, then f is Riemann integrable on $[0,1]$ and $\int_0^1 f(x)\mathrm{d}x = \lim_{n \to \infty} \Sigma_{k=1}^{n} f(\frac{k}{n})(\frac{1}{n})$.

   (d) Let $F(x,y) : \mathcal{R}^2 \longrightarrow \mathcal{R}$ and $f : \mathcal{R} \longrightarrow \mathcal{R}$ be defined by $f(t) = F(2t,t)$. If the partial derivatives of F, F_x and F_y, exist and are continuous, then f is differentiable and $f'(t) = 2F_x(2t,t) + F_y(2t,t)$.

2. (15pts). Please find $\lim_{x \to 0^+}(\cos x - 1)\ln x$. Show your work.

3. (15pts). Please find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, above the x-y plane, and between the two cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 2$. Show your work.

4. Let $V = \{p(t) : p(t) = a_0 + a_1 t$ defined on $[-1,1]$ for some $a_0, a_1 \in \mathcal{R}\}$ and $W = \{q(t) : q(t) = b_0 + b_1 t + b_2 t^2$ defined on $[-1,1]$ for some $b_0, b_1, b_2 \in \mathcal{R}\}$. Note V and W are vector spaces and having natural bases $S = \{p_1(t) = 1, p_2(t) = t\}$ and $T = \{q_1(t) = 1, q_2(t) = t, q_3(t) = t^2\}$ respectively. Consider the linear transformation $L : V \longrightarrow W$ defined by $L(p(t))(t) = \int_{-1}^{t} p(x)\mathrm{d}x$.

   (a) (12pts). Please find $[L(p_1(t))]_T$ and $[L(p_2(t))]_T$, the coordinate vectors of $L(p_1(t))$ and $L(p_2(t))$ with respect to the ordered basis T. Show your work.

   (b) (6pts). Please find the matrix representing L with respective to the bases S and T.

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　　　　　二、試題請隨卷繳交。

5. Consider this quadratic function of two variables $f(x, y) = x^2 + 6xy + y^2 - 14x - 10y$.

   (a) (10pts). Please find $x_0, y_0, c \in \mathcal{R}$ and a 2×2 symmetric matrix A such that $f(x, y) = \begin{bmatrix} x - x_0 & y - y_0 \end{bmatrix} A \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix} + c$. Show your work.

   (b) (10pts). Is the matrix A in (a) positive definite, negative definite, or neither p.d. nor p.d.? Give your reason.

6. Let V be the subspace of $\mathcal{R}^4$ with basis $S = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$, where $\mathbf{v_1} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \end{bmatrix}$,

   $\mathbf{v_2} = \begin{bmatrix} 3 \\ -1 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{v_3} = \begin{bmatrix} 2 \\ -1 \\ -2 \\ -3 \end{bmatrix}$. Also let $\mathbf{w} = \begin{bmatrix} -1 \\ -3 \\ 2 \\ 5 \end{bmatrix}$.

   (a) (10pts). Please apply the Gram-Schmidt process on S to generate an orthonormal basis for V. Show your work.

   (b) (10pts). Please find $\mathbf{v} \in V$ and $\mathbf{u} \in V^{\perp}$ such that $\mathbf{w} = \mathbf{v} + \mathbf{u}$. Show your work.