科 目	微積分	適 用 系 所	應用數學系A組、B組	時 間	100 分鐘

一、選擇題，每題4分。

1. Let $f, g : \Re \to \Re$ be discontinuous at $x = 0$. Which of the following functions is discontinuous at $x = 0$? (A) $f + g$ (B) $2f$ (C) $f \cdot g$ (D) f^2.

2. If $f(a) = 2$, $f'(a) = 1$ and $f''(a) = 3$
   then $(\frac{1}{f})''(a) =$ (A) $-\frac{3}{4}$ (B) $\frac{3}{4}$ (C) $-\frac{1}{2}$ (D) $\frac{1}{2}$.

3. $\lim\limits_{x \to 0} \dfrac{\int_2^{2+x} \sqrt{1+t^3}\, dt}{x} =$ (A) 0 (B) 1 (C) 2 (D) 3.

4. If $\alpha > 0$ and $\beta > 0$ then $\lim\limits_{x \to \infty} \dfrac{(\ln x)^\alpha}{x^\beta} =$ (A) 0 (B) $\frac{\alpha}{\beta}$ (C) 1 (D) ∞.

5. Which of the following statements is wrong?
   (A) $\int_{-1}^{2} \frac{1}{x^3}\, dx = \frac{3}{8}$ (B) $\int_{-1}^{2} \frac{x^2}{\sqrt{x^3+1}}\, dx = 2$ (C) $\int_{-1}^{1} \sqrt{1-x^2}\, dx = \frac{\pi}{2}$ (D) $\int_{-1}^{1} \sin^3 x\, dx = 0$.

6. $\int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} xy^2\, dx\, dy =$ (A) $\frac{4}{15}$ (B) $\frac{2}{15}$ (C) $\frac{1}{15}$ (D) 0.

7. Which of the following series is divergent?
   (A) $\sum\limits_{n=1}^{\infty} \dfrac{2n+3}{\sqrt{n^5+4n^3+2n}}$ (B) $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n}{n}$ (C) $\sum\limits_{n=1}^{\infty} n e^{-n}$ (D) $\sum\limits_{n=1}^{\infty} \sin \dfrac{1}{n}$.

8. Let $a_n > 0$ for all $n \in \mathrm{N}$ and $\sum\limits_{n=1}^{\infty} a_n$ be a convergent series. Which of the following series is possibly divergent?

   (A) $\sum\limits_{n=1}^{\infty} a_n^2$ (B) $\sum\limits_{n=1}^{\infty} \dfrac{a_n^2}{n}$ (C) $\sum\limits_{n=1}^{\infty} \sqrt{a_n}$ (D) $\sum\limits_{n=1}^{\infty} \dfrac{\sqrt{a_n}}{n}$.

9. The radius of convergence for $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n (x+3)^n}{n 2^n}$ is
   (A) 0 (B) $\frac{1}{2}$ (C) 2 (D) ∞.

10. If $f(x) = \tan^{-1} x$ then $f^{(100)}(0) =$ (A) 0 (B) 1 (C) $\dfrac{100!}{201}$ (D) $\dfrac{100!}{201!}$.

二、計算證明題，每題15分。

1. Suppose that $|f(x)| \le |x|^\alpha$ for all $x \in \mathfrak{R}$, where $\alpha > 1$.

   a) Find $f(0)$.

   b) Prove that f is continuous at $x = 0$.
   c) Prove that f is differentiable at $x = 0$.

2. Let $f(x) = 3x + \cos x$ for $x \in \mathfrak{R}$.
   a) Prove that f is $1-1$ on $\mathfrak{R}$. b) Find $(f^{-1})'(1)$ and $(f^{-1})''(1)$.

3. Evaluate the following integrals.

   a) $\int_0^\infty x e^{-x^2} dx$. b) $\int \sec^3 x \, dx$.

4. Let $f(x) = (1+x)^\alpha - (1 + \alpha x), x \ge 0$ and $\alpha \in (0,1)$.

   a) Prove that f is deceasing on $[0, \infty)$.

   b) Prove that $(1+x)^\alpha \le (1 + \alpha x)$ for all $x \ge 0$. (Hint: by a)

   c) Prove that $\{(1 + \frac{1}{n})^n\}$ is increasing. (Hint: by b)