國立臺灣大學101學年度碩士班招生考試試題

Do all the 6 problems.

(1) (15 pts) Determine the Galois group of $x^4 - 2$ over $\mathbb{Q}$.

(2) (20 pts) Fix a prime number p, and let $\mathbb{F}_p$ denote a field of p elements. We consider $GL(n, \mathbb{F}_p)$ the group consisting of invertible $n \times n$ matrices with entries in $\mathbb{F}_p$. How many elements are there in $GL(n, \mathbb{F}_p)$?

Moreover, if we consider $SL(n, \mathbb{F}_p)$ the group consisting of $n \times n$ matrices with determinant $= 1$ and entries in $\mathbb{F}_p$. How many elements are there in $SL(n, \mathbb{F}_p)$?

(3) (15 pts) Let $p \neq q$ be distinct prime numbers. Find an integer n with prime factorization of the form $n = pq$ such that

$$2^n \equiv 2 \pmod{n}.$$

(4) (20 pts) Consider the polynomial $f(x) = x^4 - x^3 + x^2 - x + 1$.

(a) Show that $f(x)$ is irreducible in $\mathbb{Q}[x]$.

(b) Let $\mathbb{F}_5 = \mathbb{Z}/5\mathbb{Z}$ be a field of 5 elements. Is $f(x)$ reducible or irreducible in $\mathbb{F}_5[x]$? Explain why.

(5) (15 pts) By a complex

$$0 \xrightarrow{\varphi_0} V_1 \xrightarrow{\varphi_1} V_2 \xrightarrow{\varphi_2} \dots \xrightarrow{\varphi_{n-1}} V_n \xrightarrow{\varphi_n} 0$$

of vector spaces over a field F, we mean that V_i are vector spaces over F for $i = 1, \dots, n$, φ_i are linear transformations of vector spaces for $i = 0, \dots, n$ and

$$\varphi_i \circ \varphi_{i-1} = 0$$

for $i = 1, \dots, n$.

(a) Show that $\operatorname{im}(\varphi_{i-1}) \subset \ker(\varphi_i)$.

(b) We define $H_i := \ker(\varphi_i)/\operatorname{im}(\varphi_{i-1})$. Suppose that V_i are finite dimensional vector spaces for $i = 1, \dots, n$, we can define

$$\chi_V := \sum (-1)^i \dim V_i,$$
$$\chi_H := \sum (-1)^i \dim H_i.$$

Show that $\chi_V = \chi_H$.

(6) (15 pts) Give examples of four non-isomorphic non-abelian groups of order 24. Verify your answer.

試題隨卷繳回