1. (Linear Algebra **25%**)

Given a matrix

$$A=\begin{bmatrix} 1 & 2 & -2 \\ 2 & 4 & -4 \\ -2 & -4 & 4 \end{bmatrix}$$

(a). (**10%**) Find the **eigenvalues** and their corresponding **eigenvectors** of the matrix A.

(b). (**15%**) Find the **orthonormal basis** for **each** eigenspace by using the Gram-Schmidt process.

2. (Differential Equation **30%**)

(a). (**15%**) Solve the following boundary value problem

$$\frac{\partial^2 u(x,y)}{\partial^2 x^2}+\frac{\partial^2 u(x,y)}{\partial^2 y^2}=0 \quad (0<x<a, 0<y<b)$$

$$B.C.:\begin{cases} u(0,y)=0,\ u(a,y)=T_0 \text{ (constant)} & 0<y<b \\ \left.\dfrac{\partial u(x,y)}{\partial y}\right|_{y=0}=\left.\dfrac{\partial u(x,y)}{\partial y}\right|_{y=b}=0 & 0<x<a \end{cases}$$

(b). (**15%**) Solve the following ordinary differential equation (ODE)

$$4xy''+2y'+y=0.$$

3. (Laplace Transforms **15%**)

(**15%**) Solve the following initial value problems by using the Laplace Transformation.

$$y''+3y'+2y=\begin{cases} 1 & 1<t<2 \\ 0 & otherwise \end{cases}, \quad y(0)=0, \quad y'(0)=0$$

4. (Fourier Analysis 30%)

(a). (20%) Find the Fourier **coefficients** of the period function $f(x)$ defined by

$$f(x)=\begin{cases} 0 & if\ -L<x<0 \\ E\sin\omega x & if\ \ 0<x<L \end{cases} \qquad period = 2L = \frac{2\pi}{\omega}$$

(b). (10%) Find the Fourier **transform** of the function, $f(x)=e^{-a(x-b)}$ where a and b are constants. The Fourier transform of a function, $f(x)$, is defined by

$$\Im[f(x)] = F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-j\omega x}dx .$$