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Multiple-choice questions

▷ For each of questions 1–4, pick *exactly one* of (A), (B), (C) and (D). A correct answer earns 7 points, and an incorrect one earns nothing.

1 (7 points). Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Q}$ be the set of rational numbers. Which one of the following statements is true?

(A). There exists a real number $r > 0$ satisfying $|q-r| > r/2$ for all $q \in \mathbb{Q}$.
(B). Let $2^{\mathbb{N}}$ be the power set of $\mathbb{N}$. Then there does not exist a function $f: 2^{\mathbb{N}} \to [0,1]$ with $\{f(S) \mid S \in 2^{\mathbb{N}}\} = [0,1]$.
(C). There exists a one-to-one and onto function $f: \mathbb{N} \to \mathbb{Q}$.
(D). None of the above.

2 (7 points). Which one of the following statements is true?

(A). Every tree has a Hamiltonian cycle.
(B). Every 2-regular graph is planar.
(C). There exist real numbers $x_1, x_2, \ldots, x_5$ satisfying

$$\max_{i \in \{1,2,3,4\}} |x_i - x_{i+1}| < \frac{|x_1 - x_5|}{4}.$$

(D). None of the above.

3 (7 points). Which one of the following statements is true?

(A). Every bipartite graph is planar.
(B). Let $G = (V, E)$ be a simple undirected graph without isolated vertices, where $|V| = 5$. Then G has two distinct vertices with the same degree.
(C). Every 7-element abelian group has a 4-element subgroup.
(D). None of the above.

4 (7 points). For each positive integer d and real number x, let

$$P_d(x) = \sum_{j=1}^{d} (d-2j)\, x^j,$$

for example, $P_3(x) = x - x^2 - 3x^3$. We say that a positive integer t is good if

$$\sum_{k=1}^{7t} |P_t(k)| = 6t - 1.$$

How many good positive integers are there?

(A). Zero.
(B). Two.
(C). Three.
(D). None of the above.

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Fill-in-the-blank questions

▷ For problems 1–9, fill in each blank with a real number.

1 (8 points). There are __________ subsets of the set $\{1, 2\}$.

2 (8 points). Let $a_{i,j} \in \{0, 1\}$ for all $i \in \{1, 2, 3, 4\}$ and $j \in \{1, 2, \ldots, 1000\}$. If $\sum_{j=1}^{1000} a_{i,j} < 250$ holds for all $i \in \{1, 2, 3, 4\}$, then

$$\min_{j \in \{1,2,\ldots,1000\}} \sum_{i=1}^{4} a_{i,j} = \_\_\_\_\_\_\_\_\_\_.$$

3 (8 points). It is known that the complete graph on 4 vertices has 6 edges. The complete graph on 5 vertices has __________ edges.

4 (8 points). Let $G = (V, E)$ be a simple undirected graph with $|V| = 512$. Suppose that G is 9-regular, i.e. every vertex of G has degree 9. Then

$$\frac{9 \cdot |V|}{|E|} = \_\_\_\_\_\_\_\_\_\_.$$

5 (8 points).

$$\sum_{i=1}^{30} i = \_\_\_\_\_\_\_\_\_\_.$$

6 (8 points). For each positive integer x, define $x \bmod 97$ to be the remainder of x divided by 97. For example, $204 \bmod 97$ is 10, and $100 \bmod 97$ is 3. Note that the remainder of any positive integer divided by 97 is in $\{0, 1, \ldots, 96\}$. We have

$$\frac{\sum_{i=1}^{96} 2^{(3i \bmod 97)}}{2^{99} - 8} = \_\_\_\_\_\_\_\_\_\_.$$

7 (8 points). For any positive integers x and y, denote their greatest common divisor by $\gcd(x, y)$. Then

$$\max_{k \in \{1,2,3,4,5\}} \max_{q \in \{1,2\}} \max_{r \in \{1,2,3\}} \frac{\gcd(kq + r, k)}{\gcd(k, r)} = \_\_\_\_\_\_\_\_\_\_.$$

8 (8 points). If α and β are two distinct complex numbers satisfying

$$\alpha^2 + \alpha + 3 = \beta^2 + \beta + 3 = 0,$$

then

$$(\alpha^3 + \beta^3) + (\alpha^2 + \beta^2) = \_\_\_\_\_\_\_\_\_\_.$$

9 (8 points). It is known that 541 is a prime. If x and y are integers satisfying $7^{541} = 541x + y$ and $0 \le y \le 540$, then $y = \_\_\_\_\_\_\_\_\_\_$.