1. (10%)

Alice has three children. Assume that all eight possible arrangements of "boy" and "girl" in the order of birth, $\{bbb, bbg, bgb, bgg, gbb, gbg, ggb, ggg\}$, are equally probable. You are given the information that at least one of Alice's children is a boy, and the younger child is not a girl. Condition on the given information, what is the probability that the Alice have two girls and one boy?

2. (10%)

Let X be the number of heads in 20 tosses of a fair coin. Find $\mathrm{E}\{X\}$.

3. (10%)

A random variable X has probability density function (pdf):

$$f_X(x) = \begin{cases} c(1-x^4) & , -1 \le x \le 1 \\ 0 & , \text{elsewhere} \end{cases}$$

Find c.

4. (10%)

Consider the following joint pdf:

$$f_{X,Y}(x,y) = \begin{cases} x+y & , \text{if } 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & , \text{otherwise} \end{cases}$$

Find the probability for the event "$Y \ge X + 0.5$".

5. (10%)

Let X be a random variable, where $\mathrm{E}\{X\} = 40$ and $\mathrm{VAR}\{X\} = 20$. Let $Y = aX + b$, where a and b are two constants to be designed such that $\mathrm{E}\{Y\} = 60$ and $\mathrm{VAR}\{Y\} = 5$. Find a and b.

6. (10%)

Let X and Y be two random variables with the following joint cumulative distribution function (CDF):

$F_{X,Y}(x,y) = \frac{1}{1+e^{-2x}+e^{-y}}$, $-\infty < x < \infty$, $-\infty < y < \infty$

Find the joint pdf $f_{X,Y}(x,y)$.

7. (40%)

Let X be a discrete random variable with the following probability mass function (PMF):

$$p_X(x) = \begin{cases} 1/3 & , \text{ if } x = 1 \\ 2/3 & , \text{ if } x = -2 \end{cases}$$

Let $Y = X + N$, where N is a real Gaussian random variable with zero mean and unit variance. X and N are known to be independent.

Let $\mathrm{Q}(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-t^2/2} dt$.

(a) (5%) Find $\mathrm{P}(Y < 0 | X = 1)$. Please express your answer using Q function.

(b) (5%) Find $\mathrm{P}(Y > 0 | X = -2)$. Please express your answer using Q function.

(c) (5%) What is the probability that X and Y have different signs? Please express your answer using Q function.

(d) (5%) Find $\mathrm{E}\{Y | X = 1\}$.

(e) (5%) Find $\mathrm{E}\{Y\}$.

(f) (5%) Find $\mathrm{Var}\{Y | X = -2\}$.

(g) (5%) Find $\mathrm{Var}\{Y\}$.

(h) (5%) Are X and Y jointly Gaussian? Please explain your answer.