1. (15%) For the following sampled data of a function $y=f(x)$

x	0	1	2	3	4
y	3	3	2	4	4

Please estimate $f(3.5)$ by using linear and quadratic least-squares-error criterion, respectively, and tell me the optimality between them.

2. (10%) Let R^3 have the Euclidean inner product. The subspace of R^3 spanned by the vectors $\mathbf{u}_1 = (\frac{4}{5},0,\frac{-3}{5})$ and $\mathbf{u}_2 = (0,1,0)$ is a plane passing through the origin. Express $\mathbf{w} = (1,2,3)$ in the form $\mathbf{w} = \mathbf{w}_1 + \mathbf{w}_2$, where $\mathbf{w}_1$ lies in the plane and $\mathbf{w}_2$ is perpendicular to the plane.

3. (15%) (a) Show that if $\mathbf{v}$ is any $n\times 1$ matrix and I is the $n\times n$ identity matrix, then $I - \mathbf{v}\mathbf{v}^T$ is orthogonally diagonalizable.

(b) Find a matrix P that orthogonally diagonalizes $I - \mathbf{v}\mathbf{v}^T$ if

$$\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

4. (10%) Let λ_1 and λ_2 be distinct eigenvalues of A. Let $\mathbf{x}$ be an eigenvector of A belonging to λ_1 and let $\mathbf{y}$ be an eigenvector of A^T belonging to λ_2. Show that $\mathbf{x}$ and $\mathbf{y}$ are orthogonal.

5. (10%) Let $A = [a_{ij}]$ be a 2x2 matrix with $a_{22} \neq 0$. The matrix A can be factored into a product of the form

$$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a_{11} & 0 \\ a_{21} & y \end{bmatrix}$$

What are the values of x and y?

6. Let B be a 3x3 matrix with three column vectors $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{b}_3$. Suppose that

$$2\,\mathbf{b}_1 + 4\,\mathbf{b}_2 + 6\,\mathbf{b}_3 = \mathbf{0}$$

(a) (5%) Will the corresponding homogeneous system have trivial solution?

(b) (5%) Is B singular or nonsingular? (Explain your answer)

7. In coding a message, a blank space was represented by 0, an 'A' by 1, a 'B' by 2, a 'C' by 3, and so on. The message was transformed using the matrix

$$C = \begin{bmatrix} -1 & 1 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ 2 & -1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

and sent as

0, 0, -18, 21, 12, 8, -16, 11

(a) (5%) Find the determinant of C?

(b) (10%) What was the message?

8. In $\mathbf{P}_3$, there are two ordered bases $p_1 = [x\text{-}1, x^2\text{-}1]$ and $p_2 = [x^2+2x\text{-}3, 4x^2\text{-}x\text{-}3]$

(a) (10%) Find the transition matrix D corresponding to the change coordinates from p_1 to p_2.

(b) (5%) Find the nullity of the matrix D.