科目： 線性代數　　　　系所組： 數學

1. Suppose that the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & -2 & 0 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

is the reduced row echelon form of the matrix A. Denote $A^{(j)}$ the j^{th} column of the matrix A.

(a) (4 Points) Determine $A^{(5)}$ if $A^{(1)} = \begin{pmatrix} 3 \\ 1 \\ 2 \\ 7 \end{pmatrix}$, $A^{(2)} = \begin{pmatrix} -1 \\ -1 \\ -3 \\ -2 \end{pmatrix}$ and $A^{(4)} = \begin{pmatrix} 4 \\ 3 \\ 9 \\ 8 \end{pmatrix}$.

(b) (4 Points) Determine a basis for the subspace $W = \{x \in \mathbb{R}^5 : Ax = 0\}$ of $\mathbb{R}^5$.

(c) (6 Points) Find the orthogonal projection of $u = (1, -2, 0, 1, 0)$ on W.

(d) (8 Points) Find a basis for the orthogonal complement of W.

2. Let $M_{n\times n}(F)$ be the vector space of all $n \times n$ matrices from the field F. Let $W_1 = \{A \in M_{n\times n}(F) : A^t = A\}$ and $W_2 = \{A \in M_{n\times n}(F) : A^t = -A\}$. Define $T : M_{n\times n}(F) \to W_1$ by $T(A) = A + A^t$

(a) (8 Points) Determine $\dim(W_1)$ and $\dim(W_2)$.

(b) (8 Points) Is T invertible? Why?

3. Let A be a 5×5 matrix with the characteristic polynomial

$$f(t) = -t^5 + 2t^4 - t^2 - 2.$$

(a) (8 Points) Determine $\text{rank}(A)$.

(b) (8 Points) Calculate the determinant of the matrix $A^4 - 2A^3 + A$.

4. (30 Points) Prove or disprove each of the following statements.

(a) Let W_1 and W_2 be subspaces of a vector space V. Then $W_1 \cup W_2$ is a subspace of V.

(b) Let V and W be vector spaces and $T : V \to W$ be a linear transformation. If $\dim(V) > \dim(W)$, then T is one-to-one.

(c) Let $Ax = b$ be a system of m linear equations in n unknowns. If $\text{rank}(A) = m$, then $Ax = b$ has a solution.

(d) If A is an $n \times n$ matrix such that $A^2 = I_n$, the $n \times n$ identity matrix, then $A = I_n$ or $A = -I_n$.

(e) Let T be a self-adjoint linear operator on a finite-dimensional inner product space V. Then every eigenvalue of T is real.

5. (16 Points) Let W be a finite-dimensional subspace of an inner product space V and let $x \in V \backslash W$. Prove that there exist unique vectors $u \in W$ and $y \in W^{\perp}$ such that $x = u + y$.