國立臺灣師範大學 105 學年度碩士班招生考試試題

科目：高等微積分　　　　　　　　　適用系所：數學系

注意：1.本試題共 2 頁，請依序在答案卷上作答，並標明題號，不必抄題。2.答案必須寫在指定作答區內，否則依規定扣分。

1. (10 points) Some people say that $0.99999\ldots < 1$. What do you think about this?

2. (10 points) Suppose that $a_k \neq 0$ and that $\lim_{k\to\infty} \frac{\log(1/|a_k|)}{\log k} > 1$. Show that $\sum_{k=1}^{\infty} a_k$ converges absolutely.

3. (10 points) Suppose f is a continuous function defined on an interval I. Let $J = f(I)$. Show that J is also an interval.

4. (10 points) Bob says that $\int e^{x^2}\,\mathrm{d}x$ does not exist. Patrick says that e^{x^2} is not integrable. What do you say?

5. (10 points) Suppose that $0 < \alpha \leq 1$. Show that $(1+x)^\alpha \leq 1 + \alpha x$ for all $x \in [-1, \infty)$.

6. (10 points) Suppose that K is compact in $\mathbb{R}^n$ and that for every $x \in K$ there is an $r = r(x) > 0$ such that $B_r(x) \cap K = \{x\}$. Prove that K is a finite set.
   (*Notation.* $B_r(x) := \{p \in \mathbb{R}^n \mid d(x, p) < r\}$.)

7. (10 points) Let g_1, g_2 be continuous functions from $\mathbb{R}^2 \to \mathbb{R}$. Define $f: \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x, y) = \int_0^x g_1(t, 0)\,\mathrm{d}t + \int_0^y g_2(x, t)\,\mathrm{d}t.$$

   (a) Show that $\dfrac{\partial}{\partial y} f(x, y) = g_2(x, y)$.

   (b) How should f be defined so that

$$\frac{\partial}{\partial x} f(x, y) = g_1(x, y) \quad \text{and} \quad \frac{\partial}{\partial y} f(x, y) = g_2(x, y)?$$

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8. (10 points)

   (a) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function. Show that f cannot be a one-to-one function.

   (b) Let g be a real-valued differentiable function on some open set U in $\mathbb{R}^2$, and $p \in U$. If the gradient $\nabla g(p)$ is not the zero vector, show that g does not achieve an extremum at p in U.

9. (10 points) Denote by $\overline{X}$ the *closure* of a subset X in $\mathbb{R}^n$. Given that A, B, C are three non-empty subsets of $\mathbb{R}^n$, show that

$$\overline{A \cup B \cup C} = \overline{A} \cup \overline{B} \cup \overline{C}.$$

10. (10 points) Let B be the closed unit ball in $\mathbb{R}^3$, that is,

$$B := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \le 1\},$$

and denote the volume element in $\mathbb{R}^3$ by dV. Compute the definite integral

$$\int_B x^2 \, dV.$$

(試題結束)