Notice: You *must* show all your *work* in order to receive full credit.

(1) (20 points) Show that if $\mathbb{R}^n = W_1 \cup W_2 \cup \cdots \cup W_k \cup \cdots$, where each W_k is a subspace, then $\mathbb{R}^n = W_i$ holds for some i.

(2) (20 points) Let a, b, c, d, e, f be real numbers such that the quadratic form $Q(x, y, z) := ax^2 + by^2 + cz^2 + 2dxy + 2eyz + 2fxz$ is positive definite. Then the region bounded by the surface $Q(x, y, z) = 1$ has volume equals

$$\frac{4\pi}{3\sqrt{abc + 2def - ae^2 - bf^2 - cd^2}}.$$

(3) (15 points) The are infinitely many t in $\mathbb{R}$ such that the vectors $(t, 2t^2, 3t^3, 4t^4)$, $(t^2, 2t^3, 3t^4, 4)$, $(t^3, 2t^4, 3t, 4t^2)$, $(t^4, 2t, 3t^2, 4t^3)$ form a basis of $\mathbb{R}^4$.

(4) (15 points) Determine all values of $a, b, c, d, e, f \in \mathbb{R}$ such that the matrix $A := \begin{pmatrix} 1 & a & b & c \\ 0 & 1 & d & e \\ 0 & 0 & 2 & f \\ 0 & 0 & 0 & 2 \end{pmatrix}$ is *not* diagonalizable.

(5) (10 points) If a 5×5 matrix $A \in M_5(\mathbb{R})$ satisfies $A^7 = I_5$ (the identity matrix), then 1 is an eigenvalue of A.

(6) Prove or disprove the following statements (10 points for each).

(a) If $\psi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a linear transformation with the null space (kernel) of dimension $n - 1$, then there exists some $v \in \mathbb{R}^n$ and a non-zero $\lambda \in \mathbb{R}$ such that $\psi(v) = \lambda \cdot v$.

(b) If W_1 and W_2 are 8-dimensional subspaces of $\mathbb{R}^{10}$, then there exist $a_1, a_2, \cdots, a_{10}, b_1, b_2, \cdots, b_{10}, c_1, c_2, \cdots, c_{10}, d_1, d_2, \cdots, d_{10}$ in $\mathbb{R}$ such that the intersection $W_1 \cap W_2$ is the set of all vectors $(x_1, ..., x_{10})$ with $x_1, ..., x_{10}$ a solution to the system of equations

$$\begin{cases} a_1x_1 + a_2x_2 + \cdots + a_ix_i + \cdots + a_{10}x_{10} &= 0 \\ b_1x_1 + b_2x_2 + \cdots + b_ix_i + \cdots + b_{10}x_{10} &= 0 \\ c_1x_1 + c_2x_2 + \cdots + c_ix_i + \cdots + c_{10}x_{10} &= 0 \\ d_1x_1 + d_2x_2 + \cdots + d_ix_i + \cdots + d_{10}x_{10} &= 0. \end{cases}$$

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