國立臺灣師範大學 106 學年度碩士班招生考試試題

科目：**線性代數與代數**　　　　　　　　　　適用系所：**數學系**

注意：1.本試題共 2 頁，請依序在答案卷上作答，並標明題號，不必抄題。2.答案必須寫在指定作答區內，否則依規定扣分。

Part I : Linear Algebra

1. (20 points) Mark each of the following Ture or False.

   (a) Every vector space with a nonzero vector has at least two distinct subspaces.

   (b) Two subspaces of a vector space V may have empty intersection.

   (c) If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ generates a vactor space V, then each $\mathbf{v} \in V$ is a unique linear combination of the vectors in this set.

   (d) All vector spaces having a basis are finitely generated.

   (e) Any two bases in a finite-dimensional vector space V have the same number of elements.

   (f) If $B = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ is a basis for $\mathbb{R}^n$ and T and T' are linear transformations mapping $\mathbb{R}^n$ into $\mathbb{R}^m$, then $T(\mathbf{x}) = T'(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$ if and only if $T(\mathbf{b}_i) = T'(\mathbf{b}_i)$ for $i = 1, 2, \ldots, n$.

   (g) Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a basis for a vector space V. If $\mathbf{w}$ is not in $\text{sp}(\mathbf{v}_1, \mathbf{v}_2)$, then $\text{sp}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{w}) = \text{sp}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$.

   (h) Let A be a 3×4 matrix. If $A\mathbf{x} = \mathbf{b}$ is consistent for all $\mathbf{b}$, then the column space of A is $\mathbb{R}^3$.

   (i) Let A and B be square matrices and B is diagonal, then $AB = BA$.

   (j) $\left\{ \begin{bmatrix} x \\ 2x \\ 0 \end{bmatrix} \middle| \, x \geq 0 \right\}$ is a subspace of $\mathbb{R}^3$.

2. (14 points) Let $A = \begin{bmatrix} 1 & 12 & 12 & 4 \\ 2 & 6 & -1 & 1 \\ 3 & 7 & 4 & 2 \\ 4 & 9 & 8 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \\ 1 & 5 & 25 & 125 \end{bmatrix}$.
   Find the following values: $\det(A)$, $\det(B)$, $\det(A+A)$, $\det(A^TA)$, $\det(AB^{-1})$, $\det(A^5)$, $\det(\text{adj}(A))$.

3. (5 points) Let A be a matrix such that A^2 is invertible. Prove that A is invertible.

4. (5 points) Prove the similar matrices have the same eigenvalues with the same geometric multiplicity.

5. (6 points) Let $T : P_3 \to P_2$ be defined by $T(f(x)) = f'(2x-1)$ and let $B = (x^3, x^2, x, 1)$ and $B' = (x^2, x, 1)$ be ordered bases for P_3 and P_2, respectively.

   (a) Find $T(x^2 + x - 1)$.

   (b) Find $[T]_B^{B'}$, which is the matrix representation of T relative to B, B'.

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Part II : Algebra

6. (10 points) Let G be a finite group of order 85. Then

   (a) Find the numbers of the Sylow 5-subgroup and Sylow 17-subgroup of G.

   (b) Show that G is a direct product of two normal subgroups of G.

7. (10 points) In integral set $\mathbb{Z}$. We define $a * b$ to be

$$a * b = \begin{cases} a + b & \text{if } a \text{ is even,} \\ a - b & \text{if } a \text{ is odd.} \end{cases}$$

   Show that $\langle \mathbb{Z}, * \rangle$ forms a non-abelian group.

8. (10 points) If M, N are normal subgroups of G, then prove that $M \cap N$ is normal subgroup of G.

9. (10 points) Show that if $1 - ab$ is an unit in a ring R, then so is $1 - ba$.

10. (10 points) Show that (19) is a prime (maximal) ideal in $\mathbb{Z}[i]$.

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