國立臺灣師範大學 106 學年度碩士班招生考試試題

科目：**數學基礎**　　適用系所：**資訊工程學系**

注意：1.本試題共 4 頁，請依序在答案卷上作答，並標明題號，不必抄題。2.答案必須寫在指定作答區內，否則依規定扣分。

Notations (You may skip over this part):

- We work with column vectors and denote the set of all column vectors with n component by $\Re^n$. For example, $\boldsymbol{v} = \begin{bmatrix}1\\2\\3\end{bmatrix} \in \Re^3$.
- Let S_1 and S_2 be subsets of $\Re^n$ and $\Re^m$, respectively. A **function** f from S_1 to S_2, written $f: S_1 \to S_2$, is a rule that assigns to each vector $\boldsymbol{v}$ in S_1 a unique vector $f(\boldsymbol{v})$ in S_2.
- Let A be an $m\times n$ matrix. The function T_A: $\Re^n \to \Re^m$ defined by $T_A(\boldsymbol{x})=A\boldsymbol{x}$ for all $\boldsymbol{x}$ in $\Re^n$ is called the **matrix transformation induced** by A.
- Let T be a linear operator on $\Re^n$ and $\mathcal{B} = \{\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_n\}$ be a basis for $\Re^n$. The matrix $[[T(\boldsymbol{b}_1)]_{\mathcal{B}} \quad [T(\boldsymbol{b}_2)]_{\mathcal{B}} \quad \cdots \quad [T(\boldsymbol{b}_n)]_{\mathcal{B}}]$ is called the **matrix representation of T with respect to** $\mathcal{B}$, or the **$\mathcal{B}$-matrix of T.** It is denoted by $[T]_{\mathcal{B}}$.
- An $n\times n$ matrix is an **orthogonal matrix** (or **orthogonal**) if its columns form an *orthonormal* basis for $\Re^n$.
- A linear operator on $\Re^n$ is called an **orthogonal operator** (or **orthogonal**) if its standard matrix is an orthogonal matrix.

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1. (10 分) Suppose that the reduced row echelon form of matrix A is

$R = \begin{bmatrix}1 & -3 & 0 & 5 & 3\\0 & 0 & 1 & 2 & -2\\0 & 0 & 0 & 0 & 0\end{bmatrix}$. Determine A if the first and third columns of A are

$\boldsymbol{a_1} = \begin{bmatrix}1\\-1\\2\end{bmatrix}$ and $\boldsymbol{a_3} = \begin{bmatrix}2\\0\\-1\end{bmatrix}$, respectively.

2. (10 分) Let A be the standard matrix of a linear transformation T: $\Re^5 \to \Re^8$. If the range of T has dimension 3, determine the dimension of each of the following subspace:

(a) Col A

(b) Null A

(c) Row A

(d) Null A^T

(e) The null space of T.

(Note: Col A denotes the column space of matrix A, the span of column vectors of A. The transpose of matrix A is denoted by A^T.)

3. (10 分) Determine an explicit description of $T(\boldsymbol{x})$ using the given basis $\mathcal{B}$ and the matrix representation of T respective to $\mathcal{B}$.

$$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 3 & 2 \\ 2 & 1 & 2 \end{bmatrix} \text{ and } \mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

(Hint: *It needs to find the standard matrix of T.*)

4. (8 分) Let $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & c & 0 \\ 1 & 0 & -2 \end{bmatrix}$ and its characteristic polynomial be

$\det(A - \lambda I_3) = -(\lambda - c)(\lambda + 2)(\lambda - 3)$. Determine all values of the scalar c for which A is not diagonalizable.

5. (12 分) Find an orthogonal operator T on $\Re^3$ such that $T(\boldsymbol{v})=\boldsymbol{w}$ where

$$\boldsymbol{v} = \frac{1}{\sqrt{10}} \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \text{ and } \boldsymbol{w} = \frac{1}{\sqrt{5}} \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}.$$

6. (8 分) The following questions relate to inhabitants of the island of knights and knaves created by Smullyan, where knights always tell the truth and knaves always lie. You encounter two people, A and B. Determine, if possible what A and B are if they address you in the ways described. If you cannot determine what these two people are, can you draw any conclusion? That is, please answer "A is a knight (or knave)," "B is a knight (or knave)," or "I draw no conclusions about A (or B)."

(1) A says "The two of us are both knights" and B says "A is a knave."

(2) A says "At least one of us is a knave" and B says nothing.

(3) Both A and B say "I am a knight."

(4) A says "I am a knave or B is a knight" and B says nothing.

7. (8 分)

(1) If the domain consists of all integers, determine the truth value of the statement $\forall n(3n \le 4n)$.

(2) Suppose that a and b are integers, $a \equiv 4$ (**mod** 17), and $b \equiv 9$ (**mod** 17). Find the integer c with $0 \le c \le 16$ such that $c \equiv 2a + 3b$ (**mod** 17).

(3) The value of the Euler φ-function at the positive integer n is defined to be the number of positive integers less than or equal to n that are relatively prime to n. Find the value of $\varphi(10)$.

(4) Draw the graph represented by the given adjacency matrix

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 0 \\ 0 & 2 & 2 \end{bmatrix}.$$

8. (8 分)

(1) Show that if five integers are selected from the first eight positive integers, there must be a pair of these integers with a sum equal to 9.

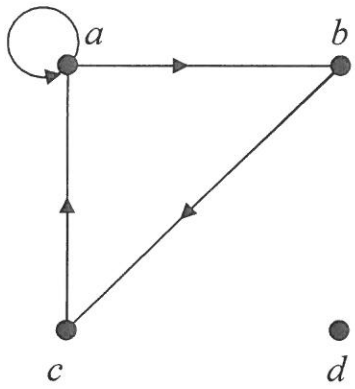
(2) Is the conclusion in part (1) true if four integers are selected rather than five?

9. (8 分)

(1) Find a matrix A such that

$$\begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 1 \\ 4 & 0 & 3 \end{bmatrix} A = \begin{bmatrix} 7 & 1 & 3 \\ 1 & 0 & 3 \\ -1 & -3 & 7 \end{bmatrix}.$$

(2) Draw the directed graph of the reflexive closure of the relation with the directed graph shown as follows.



10. (6 分) Find all solutions to the system of congruences $x \equiv 1$ (**mod** 2), $x \equiv 2$ (**mod** 3), $x \equiv 3$ (**mod** 5), and $x \equiv 4$ (**mod** 11).

11. (6 分) For the generating function $x^3/(1+3x)$, give a closed formula for the sequence it determines.

12. (6 分) Show that the function $f(x) = |x|$ from the set of real numbers to the set of nonnegative real numbers is not invertible, but if the domain is restricted to the set of nonnegative real numbers, the resulting function is invertible.