1. 【Ordinary Differential Equations】 **(20%)**

   (a) Solve $2xyy' = y^2 - x^2$ and plot the solution. **(10%)**

   (b) Solve $x^2 y'' - xy' - 3y = x^2$ by undetermined coefficients and variation of parameters respectively. **(10%)**

2. 【Vector Calculus】 **(20%)**

   (a) Evaluate the flux integral $\iint_S \vec{F} \cdot \vec{n} \, dA$ for the given data. **(10%)**

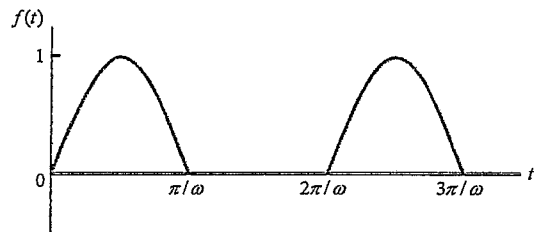 $\vec{F} = [0, \sin y, \cos z]$, S the cylinder $x = y^2$, where $0 \le y \le \pi/4$, $0 \le z \le y$.

   (b) Prove the "*Parallelogram equality*" $\left|\vec{a} + \vec{b}\right|^2 + \left|\vec{a} - \vec{b}\right|^2 = 2\left(\left|\vec{a}\right|^2 + \left|\vec{b}\right|^2\right)$. **(10%)**

3. 【Laplace Transform】 **(20%)**

   (a) Show the Laplace transform of a piecewise continuous function $f(t)$ with period p is

   $$L(f) = \frac{1}{1 - e^{-ps}} \int_0^p e^{-st} f(t) dt, \; s > 0 \quad (1) \qquad \textbf{(10\%)}$$

   (b) Using (1), find the half-wave rectification of $\sin \omega t$. **(10%)**

4. 【Fourier Analysis】 **(20%)**

   (a) Find the Fourier series of the function $f(x) = x^2$, $-1 < x < 1$ **(10%)**

   (b) Show that, $\dfrac{1}{1} + \dfrac{1}{4} + \dfrac{1}{9} + \dfrac{1}{16} + \cdots = \dfrac{\pi^2}{6}$ **(10%)**

5. 【Partial Differential Equation】 **(10%)**

   Solve the following PDE using the Method of Separation of Variables.
   $$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0, \; 0 < x < a, \; 0 < y < b$$
   boundary conditions: $T(a,y) = T(x,0) = T(x,b) = 0$, $T(0,y) = g(y)$

6. 【Residue Integration】 **(10%)**

   Evaluate $\displaystyle\int_0^\infty \frac{\cos x}{x^2 - 1} dx$