There are 7 problems in this test. No calculators are allowed. Write down detailed steps for the solution to each problem. Otherwise, no credits for that problem will be given.

1. (10) Let $x_1, x_2, \ldots, x_n$ be a sequence of n integers. A consecutive subsequence of $x_1, x_2, \ldots, x_n$ is a subsequence $x_i, x_{i+1}, \ldots, x_j$ for some i, j, $1 \le i \le j \le n$. Show that for any k, $1 \le k \le n$, there is a consecutive subsequence whose sum is divisible by k.

2. (10) Assume that a sequence of numbers is defined by $x_0 = 0$, $x_1 = 1$, and $x_n + 2x_{n-1} = 15x_{n-2}$ for $n > 1$. Find generating function for the sequence, and then find an explicit expression for x_n.

3. (10) Show that $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$. Give combinatorial explanation to the equation.

4. (30) A *bipartite graph* is a graph whose vertices can be partitioned into two subsets X and Y so that each edge has one end in X and the other end in Y. A *cycle* of G is a sequence of vertices $v_0, v_1, \ldots, v_l$ such that each vertex v_i is distinct, except $v_0 = v_l$, and v_{i-1} and v_i are adjacent for each $i = 1, 2, \ldots, l$. The *length of the cycle* is l. A *k-cube* is a graph whose vertices are binary strings of length k, for some integer $k > 0$. Two vertices are adjacent if and only if they differ in exactly one bit.

   (a) Show that every k-cube is bipartite by partitioning its vertexes into X and Y, and then show that every edge connects some vertex in X and another vertex in Y. (10)

   (b) Show that a bipartite graph has no cycles of odd length. (10)

   (c) Show that if a graph has no cycles of odd length then it is bipartite. (10)

5. (10) Suppose that 5 points are chosen in a square whose sides have length 2. Show that there must be at least two points p and q such that the distance between them is no more than $\sqrt{2}$.

6. (10) Fibonacci numbers are defined as $f_0 = 0$, $f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n > 1$. Show that f_{3k} is even, for every positive integer k.

7. (20) Let m and n be two positive integers, $m \le n$. Define $\binom{n}{m} = \frac{n!}{m!(n-m)!}$.

   (a) Show that if n is prime, then n divides $\binom{n}{i}$ for every i, $1 \le i < n$. (10)

   (b) Show that if n is composite, then n does not divide $\binom{n}{i}$ for some i, $1 \le i < n$. (10)