國立中山大學 106 學年度碩士暨碩士專班招生考試試題

科目名稱：工程數學乙【電機系碩士班乙組】 題號：431001

※本科目依簡章規定「可以」使用計算機（廠牌、功能不拘）（選擇題） 共 5 頁第 1 頁

第一題到第六題為單選題，每題四分。請選出一個最正確選項，答錯不倒扣。

1. When using the *Gaussian elimination* to solve a linear equation $A\mathrm{x} = \mathrm{b}$, elementary row operations (or multiplication by elementary matrices) are applied to the augmented matrix $[A\ \mathrm{b}]$. Actually, there are many places in linear algebra where such a technique plays its role; e.g., the null space of A can be determined by setting $\mathrm{b} = 0$. Which of the following cannot be determined by applying such a technique?
   (A) the range of A
   (B) the inverse of A (if it is nonsingular)
   (C) the QR factorization of A
   (D) the LU factorization of A (if it is square)
   (E) the determinant of A (if it is square).

2. Consider the system represented by the differential equation $\ddot{x} + \dot{x} + kx = 0$. Which of the following is true? (A) the system is critically damped when $k = 1$ (B) the system is underdamped when $k = 1/2$ (C) the system is overdamped when $k = 0.3$ (D) the damping ratio of the system is increased when increasing k (E) none of the above

3. Consider the differential equation $\ddot{x} + b\dot{x} + kx = \cos(\omega t)$ and its sinusoidal solution $x_p(t)$. Which of the following is true? (A) for a fixed b, the amplitude of $x_p(t)$ is maximized when $k = \omega^2$ (B) for a fixed b, the maximal amplitude of $x_p(t)$ is $1/b$ (C) for fixed b and k, the amplitude of $x_p(t)$ always grows as ω increases (D) the period of $x_p(t)$ is 2ω (E) the period of $x_p(t)$ depends on b and k.

4. Consider the differential equation $t\dot{x} + x = \cos(\omega t)$. Which of the following is true? (A) the general solution is $\sin(\omega t) + C$, C is a constant. (B) the particular (forced) solution is periodic with period ω (C) the solution is NOT bounded when t grows to infinity (D) If $x(\pi/\omega) = 1$, then the solution is $(\sin(\omega t) + \pi)/(\omega t)$ (E) none of the above

5. Consider a system whose dynamics is governed by a linear constant-coefficient ODE. Suppose the impulse response of this system is $\mathrm{e}^{-t} - \mathrm{e}^{-3t}$. Which of the following is true? (A) the system is 2^{nd} order (B) the system is 3^{rd} order (C) the characteristic equation of the system has a pair of complex roots (D) one root of the characteristic equation is less than -5 (E) response to a step function does NOT converge to a constant value

6. Consider the system described in Question 5 again. The unit step response of this system
   (A) has oscillatory behavior
   (B) converges to 1
   (C) converges to 1/3
   (D) is $1/4 - \mathrm{e}^{-3t}/3 + \mathrm{e}^{-t}$
   (E) is $-\mathrm{e}^{-t} + (2 + \mathrm{e}^{-3t})/3$

第七題到第十題為單選題，每題五分。請選出一個最正確選項，答錯不倒扣。

7. What is the solution to the ODE $t\dot{x} = 4t - 3x$? (A) $x(t) \equiv$ constant (B) $x(t) = -ct^3$ (C) $x(t) = ct^{-3} + t$ (D) $x(t) = c\mathrm{e}^{-3t} + 4t$ (E) none of the above

8. Consider the heat equation

$$\frac{\partial u}{\partial t}(x,t) = 4\frac{\partial^2 u}{\partial x^2}(x,t), \quad \forall 0<x<1,\ t>0$$
$$u(0,t)=u(1,t)=0 \quad \forall t>0$$
$$u(x,0)=f(x), \quad \forall 0<x<1$$

   (A) the heat equation is nonlinear
   (B) without considering the boundary condition, the general solution is $\sum_{n=1}^{\infty} c_n \sin(n\pi x)e^{-4n^2\pi^2 t}$
   (C) suppose $f(x)=7\sin(3\pi x)$. then the solution is $u(x,t)=7\sin(3\pi x)e^{-4n^2\pi^2 t}$
   (D) all of the above are true
   (E) none of the above is true

9. The *linear combination* of a set of vectors is an essential element in linear algebra. We say a set V is invariant under linear combination if the implication " $\forall n\in\mathbb{N}$ and any set of vectors $\{\mathbf{v}_1,\cdots,\mathbf{v}_n\}\subset V \Rightarrow$ the set of all linear combinations $\{c_1\mathbf{v}_1+\cdots+c_n\mathbf{v}_n | c_i\in\mathbb{R},\forall i\}\subset V$" holds. And we say a mapping L defined on a set X is invariant under linear combination if the form of linear combination is unchanged under L, or more precisely the statement "$\forall n\in\mathbb{N}$, $\forall c_i\in\mathbb{R}$, and any set of vectors $\{\mathbf{x}_1,\cdots,\mathbf{x}_n\}\subset X$, the identity $L(c_1\mathbf{x}_1+\cdots+c_n\mathbf{x}_n)=c_1L(\mathbf{x}_1)+\cdots+c_nL(\mathbf{x}_n)$ holds" is true. Which one of the following statements related to linear combination is false?
   (A) Let S be a subset of a vector space V. Then S is a subspace of V if S is invariant under linear combination.
   (B) Let $\{V_1,\cdots,V_k\}$ be a set of k subspaces of a vector space W and denote $\text{span}\{V_1,\cdots,V_k\}$ as the set of all linear combinations of the form $c_1\mathbf{v}_1+\cdots+c_n\mathbf{v}_n$, with each $\mathbf{v}_i$ chosen freely from V_i . Then $\text{span}\{V_1,\cdots,V_k\}$ is also a subspace of W with $\dim(\text{span}\{V_1,\cdots,V_k\}) = \dim(V_1)+\cdots+\dim(V_k)$
   (C) Let A and B be two matrices and denote $C:=AB$. Then each column of C is a linear combination of all columns of A , and so $rank(C)\le rank(A)$ is implied.
   (D) A mapping L between two vector spaces is a linear transformation if and only if it is invariant under linear combination.
   (E) Let $(V,\ \langle\cdot,\cdot\rangle_V)$ be an inner product space. Then $\langle\cdot,\cdot\rangle_V$ is invariant under linear combination at either one of its two arguments.

10. Let U, V, W be vector spaces and $T:U\to V$, $S:V\to W$ be linear transformations. Also denote the set of all linear transformations from V to W by $\mathcal{L}(V,W)$ (e.g. $S\in\mathcal{L}(V,W)$). Which one of the following statements is false?
   (A) The set $(\mathcal{L}(V,W),+,\bullet)$, where $(\alpha\bullet S_1+\beta\bullet S_2)(\mathbf{v})=\alpha\cdot S_1(\mathbf{v})+\beta\cdot S_2(\mathbf{v})$ for any $\mathbf{v}\in V$, $S_i\in\mathcal{L}(V,W)$, and $\alpha,\beta\in\mathbb{R}$, is also a vector space with $\dim\mathcal{L}(V,W)=\dim V+\dim W$.
   (B) The mapping $ST(\cdot)$ defined as $ST(\mathbf{u}):=S(T(\mathbf{u}))$ for any $\mathbf{u}\in U$ is also a linear transformation from U to W .
   (C) If both S and T are one-to-one, then their combination $ST(\cdot)$ is also one-to-one.
   (D) If $S:V\to W$ is invertible, then the mapping S^{-1} from W to V is also a linear transformation.
   (E) Suppose that the combination $ST(\cdot)$ is onto. Then at least one of S and T is onto.

第十一題到第十六題為多選題，每題六分。請選出所有正確選項；答題完全正確得滿分，答錯任何選項則該題以零分計，沒有倒扣。

11. The *linear independence* is without doubt one of the most important concepts in linear algebra. Which of the following statements about this concept are true?
   (A) The number of linearly independent columns of any matrix is equal to the number of linearly independent rows of the matrix.
   (B) A set of vectors $\{\mathbf{v}_1,\cdots,\mathbf{v}_n\}$ is linearly independent if there exist coefficients $c_1,\cdots,c_n$, all of them are zero, such that $c_1\mathbf{v}_1+\cdots+c_n\mathbf{v}_n=\mathbf{0}$.
   (C) The linear independence of a set of vectors is a necessary but not sufficient condition for them to be a basis of a vector space.
   (D) A square matrix is diagonalizable if and only if all its eigenvectors are linearly independent.
   (E) Let $(\lambda_i,\mathbf{x}_i)$ be the ith eigenvalue-eigenvector-pair of a square matrix. Then all $\lambda_i's$ are distinct if and only if the set $\{\mathbf{x}_1,\cdots,\mathbf{x}_n\}$ of eigenvectors is linearly independent.

12. Let $V=C[-1,1]$ be the vector space of all continuous functions defined on $[-1,1]$, and let V_e and V_o denote, respectively, the set of all even and all odd functions in V. Moreover, define an inner product $\langle f,g\rangle := \int_{-1}^{1} f(x)g(x)dx$ for any $f(\cdot),\ g(\cdot)\in V$. Which of the following statements are true?
   (A) Both V_e and V_o are subspaces of V.
   (B) $V=V_e+V_o$ and $V_e\cap V_o=\mathbf{0}$.
   (C) $V_e=V_o^{\perp}$ and $V_o=V_e^{\perp}$.
   (D) $\dim V=\dim V_e+\dim V_o$.
   (E) Denote $\text{dist}(f,V_e)$ as the distance induced from the defined inner product between any $f(\cdot)\in V$ and V_e. Then $\text{dist}\left((x+1)^2,V_e\right)=\sqrt{8/3}$.

13. Which of the following statements are true?
   (A) Given A and $\mathbf{b}$ of proper dimensions, when $\mathbf{b}\notin R(A)$ the linear equation $A\mathbf{x}=\mathbf{b}$ has no solution. However, the associated LSP (least squares problem) is always solvable and the solution is unique.
   (B) Let V be a vector space such that $V=X\oplus Y$. Then only when $X\perp Y$, i.e. the two subspaces are orthogonal, can two projection mappings, say $P:V\to X$ and $Q:V\to Y$, be defined with the complementary property $P+Q=I$, where I indicates the identity mapping on V.
   (C) (continue from (B)) The projections P and Q become orthogonal projections when $X\perp Y$. Moreover, once the bases for X and Y are chosen, their union forms a basis for V, and the two projection matrices associated with projections P and Q with respect to this set of bases are all symmetric.
   (D) Any projection mapping is a linear transformation that is definitely onto, but may not be one-to-one.
   (E) Consider the vector space $C[-1,1]$ with an inner product $\langle f,g\rangle := \int_{-1}^{1} f(x)g(x)dx$. Then the set $\{\mathbf{u}_1,\mathbf{u}_2\}$ with $\mathbf{u}_1=1/\sqrt{2}$ and $\mathbf{u}_2=(\sqrt{6}/2)x$ forms an orthonormal set in $C[-1,1]$. Moreover, the best least squares approximation to $h(x)=x^2$ by a linear function is $\hat{h}(x)=(\sqrt{2}/3)+(\sqrt{6}/4)x$.

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14. Let $E := \{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$ be a basis of $\mathbb{R}^n$ and $F := \{\mathbf{v}_1, \cdots, \mathbf{v}_n\} \subset \mathbb{R}^n$ with the property $\forall i, j = 1, \cdots, n$ $\mathbf{u}_i^T \mathbf{v}_j = 1$ when $i = j$, and $\mathbf{u}_i^T \mathbf{v}_j = 0$ when $i \neq j$. Which of the following statements are true?
   (A) F is also a basis of $\mathbb{R}^n$, and the coordinate vector of any $\mathbf{x} \in \mathbb{R}^n$ with respect to base F is $\begin{bmatrix} \mathbf{x}^T\mathbf{v}_1 \\ \vdots \\ \mathbf{x}^T\mathbf{v}_n \end{bmatrix}$.
   (B) Any $\mathbf{x} \in \mathbb{R}^n$ can be represented as $\mathbf{x} = c_1\mathbf{u}_1 + \cdots + c_n\mathbf{u}_n$ with $c_i = \mathbf{x}^T\mathbf{u}_i$ for each i.
   (C) Denote the transition matrix from base E to base F by S. Then $S(i,j) = \mathbf{u}_i^T\mathbf{u}_j$ for $i, j = 1, \cdots, n$.
   (D) The transition matrix from base F to base E can be described by $([\mathbf{v}_1]_E \cdots [\mathbf{v}_n]_E)$, where $[\mathbf{v}_i]_E$ denotes the coordinate vector of $\mathbf{v}_i$ with respect to base E.
   (E) Denote $U := [\mathbf{u}_1 \cdots \mathbf{u}_n]$ and $V := [\mathbf{v}_1 \cdots \mathbf{v}_n]$. Then $U\mathbf{x} = \lambda\mathbf{x}$ for some $\mathbf{x} \neq \mathbf{0}$ iff $(\bar{\mathbf{x}})^T V = (\bar{\lambda})^{-1}(\bar{\mathbf{x}})^T$, where the upper bar means to take the complex conjugate.

15. For any given $0 \neq \mathbf{w} \in \mathbb{R}^n$, let's consider the matrix $H_\alpha = I - \alpha(\mathbf{w}^T\mathbf{w})^{-1}\mathbf{w}\mathbf{w}^T$ parameterized by a real scalar α. Which of the following statements are true?
   (A) $H_\alpha = H_\alpha^2$, i.e. H_α is an idempotent matrix, if and only if $\alpha \in \{0, 1, 2\}$.
   (B) $H_\alpha^2 \mathbf{x} = \mathbf{x}$ for any $\mathbf{x} \in \mathbb{R}^n$ if and only if $\alpha \in \{0, 1, 2\}$.
   (C) H_α is singular if and only if $\alpha = 1$.
   (D) For any $\mathbf{x} \in \mathbb{R}^n$, $H_\alpha\mathbf{x} \in \mathbf{w}^\perp$, the orthogonal complement of $\mathbf{w}$, if and only if $\alpha = 1$.
   (E) For any $\mathbf{x} \in \mathbb{R}^n$, $\min_{\alpha\in\mathbb{R}} \|H_\alpha\mathbf{x}\|_2 = \|H_1\mathbf{x}\|_2$, i.e. $H_\alpha\mathbf{x}$ has the smallest $2-$norm if and only if $\alpha = 1$.

16. Let $A = A^H := (\bar{A})^T$ be a nonzero matrix of size $n \times n$ with its real part and imaginary part being denoted by B and C, respec 7. tively. Moreover, let be a symmetric matrix with and as its four blocks. Which of the following statements are true?
   (A) Matrix C has at least one real eigenvalues.
   (B) Let λ and μ be any two eigenvalues of B and C, respectively. Then $\lambda + \mu \neq 0$.
   (C) Any eigenvalue of A is also an eigenvalue of Ω.
   (D) Any eigenvalue of Ω is also an eigenvalue of A.
   (E) Let $\left(\lambda, \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}\right)$ be an eigenvalue-eigenvector-pair of Ω. Then $(\lambda, \mathbf{x} + i\mathbf{y})$ is an eigenvalue-eigenvector-pair of A.

第十七題到第二十題為多選題，每題五分。請選出所有正確選項；答對一個選項得一分，答錯一個選項倒扣零點五分。

17. Let $\mathcal{L}(\cdot)$ denote the Laplace transform. Which of the followings are true?
   (A) $\mathcal{L}(\alpha_1 \cdot f_1 + \alpha_2 \cdot f_2) = \alpha_1 \cdot \mathcal{L}(f_1) + \alpha_2 \cdot \mathcal{L}(f_2)$, for all $\alpha_1,\ \alpha_2 \in \mathbb{R}$ and for all functions $f_1,\ f_2$
   (B) if $\mathcal{L}(f) = F(s)$, then $\mathcal{L}(\frac{d}{dt}f) = sF(s)$
   (C) if $\mathcal{L}(f) = F(s)$, then $\mathcal{L}(e^{\alpha t} \cdot f) = F(s)/(s+\alpha)$
   (D) if $\mathcal{L}(f) = F(s)$, then $\mathcal{L}(t^2 \cdot f) = \frac{d^2}{ds^2}F(s)$
   (E) if $\mathcal{L}(f) = F(s)$ and $\mathcal{L}(g) = G(s)$, then $\mathcal{L}(f \cdot g) = (F * G)(s)$, where $*$ denotes convolution operation

18. Consider the differential equation $m\ddot{x}(t) + b\dot{x}(t) + kx(t) = u(t)$. Suppose $\frac{1}{2}t\sin(2t)$ is a solution when $u(t) = 4\cos(2t)$. Which of the followings are true?
   (A) $m = 1$
   (B) $b = 0$
   (C) $k = 8$
   (D) $2t\sin(t)$ is a solution when $u(t) = 8\cos(2t)$
   (E) $\frac{1}{2}(t - \frac{1}{2})\sin(2t-1)$ is a solution when $u(t) = 4\cos(2t-1)$

19. Consider the homogeneous system $\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} = A\begin{bmatrix}x_1 \\ x_2\end{bmatrix}$, where $A = \begin{bmatrix}a & -2 \\ 2 & 1\end{bmatrix}$
   (A) the system is linear
   (B) the system is time invariant
   (C) the system has more than one equilibrium
   (D) when $a < -4$, any initial condition results in a solution which diverges to infinity
   (E) when $-4 < a < -1$, any initial condition results in a solution which converges to the origin

20. Consider the autonomous system $\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}x_1^2 - x_2^2 \\ x_1^2 + x_2^2 - 8\end{bmatrix}$
   (A) the system is linear
   (B) the system has two equilibria
   (C) $(x_1, x_2) = (2, -2),\ (-2, -2)$ are equiliria of the system
   (D) $(x_1, x_2) = (-2, -2)$ is the only stable equilibrium of the system
   (E) $(x_1, x_2) = (2, -2)$ is a saddle equilibrium of the system.

End of Examination