國立中山大學 106 學年度碩士暨碩士專班招生考試試題

科目名稱：機率與統計【應數系碩士班甲組】　　　　　　題號：424006
※本科目依簡章規定「不可以」使用計算機(問答申論題)　　共 1 頁第 1 頁

答題時，每題須寫下題號與詳細步驟。 請依題號順序作答，不會作答題目請寫下題號並留空白。

Notation:

i.i.d.: identically independently distributed; pdf: probability density function; MLE: maximum likelihood estimator; $Exp(\theta)$ random variable means a random variable with exponential distribution with a parameter θ and its pdf is $f(x|\theta) = \frac{1}{\theta}e^{-\frac{x}{\theta}}, x > 0$. $\bar{X}$ is the sample mean.

1. (15%) $X_1, \ldots, X_n$ are i.i.d. random variables with mean μ and variance σ^2. Assume $\mu \neq 0$. What is the asymptotic distribution of

$$\sqrt{n}(\frac{1}{\bar{X}} - \frac{1}{\mu}) + \frac{\sum_{i=1}^{n} X_i^2}{n}?$$

Express your answer using μ and σ.

2. (15%) A stick of unit length is randomly cut into two pieces. Please compute the expected ratio of the length of the shorter piece to that of the longer piece.

3. (15%) Let $X_1, \ldots, X_n$ be i.i.d. $Exp(1)$ random variables. Define the random variables $Y_1, \ldots, Y_n$ by $Y_i = 1$ if $X_i \geq 3$ and $Y_i = 0$ if $X_i < 3$. Find the distribution of $\sum_{i=1}^{n} Y_i$.

4. (20%) Let $X_1, \ldots, X_n$ be $Exp(\theta_1)$ random sample and $Y_1, \ldots, Y_n$ $Exp(\theta_2)$ random sample. $X's$ and $Y's$ are independent.
   Find the likelihood ratio test of $H_0 : \theta_1 = \theta_2$ against $H_a : \theta_1 \neq \theta_2$ with significance level α (You have to specify the test statistic and the rejection region).

5. (20%) $X_1, \ldots, X_n$ are i.i.d. Bernoulli(p) random variables. Let $q = P(X_1 = 1, X_2 = 1)$. Find the uniformly minimal-variance unbiased estimator (UMVUE) of q.

6. (15%) Let $X_1, \ldots, X_n$ be a random sample from a continuous distribution F (no ties). The distribution of F has mean μ and variance σ^2. Let $X_1^*, \ldots, X_n^*$ be the sample with replacement randomly chosen from $X_1, \ldots, X_n$. Let $\bar{X}^* = \frac{\sum_{i=1}^{n} X_i^*}{n}$. Please Calculate $Var(\bar{X}^*)$ (Express your answer using σ and n).