類組：電機類　科目：工程數學D(3006)　

※請在答案卡內作答

- 本測驗試題為多選題（答案可能有一個或多個），請選出所有正確或最適當的答案，並請用2B鉛筆作答於答案卡。
- 共二十題，每題五分。每題ABCDE每一選項單獨計分；每一選項的個別分數為一分，答錯倒扣一分。

Notation: In the following questions, underlined letters such as $\underline{a}, \underline{b}$, etc. denote column vectors of proper length; boldface letters such as $\mathbf{A}, \mathbf{B}$, etc. denote matrices of proper size; $\mathbf{A}^\top$ means the transpose of matrix $\mathbf{A}$. $\mathbf{I}_n$ is the $(n \times n)$ identity matrix. $\|\underline{a}\|$ means the Euclidean norm of vector $\underline{a}$. $\mathbb{R}$ is the usual set of all real numbers; $\mathbb{C}$ is the usual set of all complex numbers. By $\mathbf{A} \in \mathbb{R}^{m \times n}$ we mean $\mathbf{A}$ is an $m \times n$ real-valued matrix. $u(x)$ is unit-step function defined as $u(x) = 1$ if $x \geq 0$ and $u(x) = 0$ if $x < 0$; $\star$ is the convolution operator; $\mathscr{L} : f(x) \mapsto F(s)$ and $\mathscr{L}^{-1} : F(s) \mapsto f(x)$ denote the unilateral Laplace and inverse Laplace transforms for $x \geq 0$, respectively.

一、 Let $\mathbf{H} = \mathbf{I}_n - 2\underline{u}\,\underline{u}^\top$, where $\underline{u} \in \mathbb{R}^n$, $n \geq 2$ and $\|\underline{u}\| = 1$. Which of the following statements is/are true?

(A) The matrix $\mathbf{H}$ is both symmetric and orthogonal.

(B) Both 1 and -1 are eigenvalues of $\mathbf{H}$.

(C) $\det(\mathbf{H}) = 1$.

(D) $\text{Trace}(\mathbf{H}) = n - 2$.

(E) None of the above.

二、 Two square matrices $\mathbf{A}$ and $\mathbf{B}$ are similar, denoted by $\mathbf{A} \sim \mathbf{B}$, if $\mathbf{B} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ for some nonsingular matrix $\mathbf{P}$. Which of the following statements is/are true?

(A) Two similar matrices always have the same set of eigenvalues, including multiplicity.

(B) Two $n \times n$ matrices having the same set of eigenvalues, including multiplicity, are similar.

(C) Any two square matrices with the same trace and determinant are similar.

(D) If $\mathbf{A} \sim \mathbf{B}$, then $p(\mathbf{A}) \sim p(\mathbf{B})$ for any polynomial $p(x)$.

(E) None of the above.

三、 Let $\mathbf{A} = \underline{x}\,\underline{y}^\top$, where $\underline{x}$ and $\underline{y}$ are two nonzero vectors of $\mathbb{R}^n$, $n > 1$. Which of the following statements is/are true?

(A) $\text{rank}(\mathbf{A}) = 1$ and the range space of $\mathbf{A}$ is $\text{Span}\{\underline{y}\}$.

(B) $\text{nullity}(\mathbf{A}) = 2$ and the null space of $\mathbf{A}$ is $\text{Span}\{\underline{x}, \underline{y}\}$.

(C) $\text{Trace}(\mathbf{A}) = 1$ and $\det(\mathbf{A}) = 0$

(D) $\mathbf{A}$ is always diagonalizable.

(E) None of the above.

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四、 Let $\mathbf{A} = \underline{x}\,\underline{y}^\top + \underline{y}\,\underline{x}^\top$, where $\underline{x}$ and $\underline{y}$ are two nonzero orthonormal vectors of $\mathbb{R}^n$ and $n > 2$. Which of the following statements is/are true?

(A) Both $\underline{x}$ and $\underline{y}$ are eigenvectors of $\mathbf{A}$

(B) $\text{Trace}(\mathbf{A}) = 1$ and $\det(\mathbf{A}) = 0$

(C) $\mathbf{A}$ is not diagonalizable.

(D) The least square solution of $\mathbf{A}\underline{z} = \underline{b}$, where $\underline{b}$ is a vector in $\mathbb{R}^n$, is $(\underline{b}^\top \underline{x})\underline{x} + (\underline{b}^\top \underline{y})\underline{y}$.

(E) None of the above.

五、 Let $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n\times n}$, and $\{\underline{u}_1, \ldots, \underline{u}_n\}$ be an orthonormal basis for $\mathbb{R}^n$. It is known that $\langle \mathbf{A}, \mathbf{B}\rangle = \text{Trace}(\mathbf{A}^\top \mathbf{B})$ is an inner product. We denote $\mathbf{A} \perp \mathbf{B}$ if $\langle \mathbf{A}, \mathbf{B}\rangle = 0$. Which of the following statements is/are true?

(A) Let $\underline{x}, \underline{y}, \underline{w}, \underline{z}$ be four nonzero vectors of $\mathbb{R}^n$. Then $\underline{w}\,\underline{z}^\top \perp \underline{x}\,\underline{y}^\top$ if and only if $\underline{w} \perp \underline{x}$ and $\underline{z} \perp \underline{y}$.

(B) The set $\mathcal{B}_1 := \left\{\underline{u}_i\,\underline{u}_j^\top : i, j = 1, \ldots, n\right\}$ is an orthonormal basis for $\mathbb{R}^{n\times n}$.

(C) The set

$$\mathcal{B}_2 = \left\{\underline{u}_i\,\underline{u}_i^\top + \frac{\underline{u}_i\,\underline{u}_j^\top + \underline{u}_j\,\underline{u}_i^\top}{\sqrt{2}} \;:\; 1 \le i < j \le n\right\}$$

is an orthonormal basis for the real vector space $\mathcal{S}_1 = \left\{\mathbf{A} \in \mathbb{R}^{n\times n} : \mathbf{A} = \mathbf{A}^\top\right\}$.

(D) The set

$$\mathcal{B}_3 = \left\{\frac{\underline{u}_i\,\underline{u}_j^\top - \underline{u}_j\,\underline{u}_i^\top}{\sqrt{2}} \;:\; 1 \le i < j \le n\right\}$$

is an orthonormal basis for the real vector space $\mathcal{S}_2 = \left\{\mathbf{A} \in \mathbb{R}^{n\times n} : \mathbf{A} = -\mathbf{A}^\top\right\}$.

(E) None of the above.

六、 The system of linear equations $\mathbf{A}\underline{x} = \underline{b}$ has

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad \underline{b} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

Which of the following statements is/are true?

(A) $\text{rank}(\mathbf{A}) + \text{nullity}(\mathbf{A}) = 3$

(B) $\mathbf{A}^\top \mathbf{A}$ is a symmetric 2×2 matrix

(C) The nullspace of $\mathbf{A}$ has two linearly independent vectors.

(D) $\mathbf{A}^\top \mathbf{A}$ is an invertible matrix.

(E) None of the above.

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七、 Continued from Problem 六, which of the following statements is/are true?

(A) The matrix $\mathbf{A}$ has the same row space as the following matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

(B) $\det(\mathbf{A}\mathbf{A}^\top) = 0$

(C) Let $\underline{p}$ be the projected vector of $\underline{b}$ onto the column space of $\mathbf{A}$. The Euclidean distance between $\underline{b}$ and $\underline{p}$ is zero.

(D) There exists a (3×2) matrix $\mathbf{C}$ such that $\text{rank}(\mathbf{CA}) = 3$.

(E) None of the above.

八、 Continued from Problem 六, let $\mathbf{B}_1$ be a (2×2) matrix and consider the system $\mathbf{B}_1\mathbf{A}\underline{x} = \mathbf{B}_1\underline{b}$. It is known that

$$\mathbf{B}_1\underline{b} = \begin{bmatrix} 5 \\ 0 \end{bmatrix} \text{ and } \underline{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Which of the following vectors can be column vectors for $\mathbf{B}_1$?

(A) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

(B) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

(C) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(D) $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

(E) None of the above.

參考用

九、 Given the matrices

$$\mathbf{M}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{M}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \mathbf{M}_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

which of the following statements is/are true?

(A) $\mathbf{M}_1$, $\mathbf{M}_2$ and $\mathbf{M}_3$ are linearly independent over $\mathbb{R}$ in $\mathbb{R}^{2\times 2}$.

(B) The span of $\{\mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3\}$ is the set of all (2×2) real matrices

(C) The set of all Hermitian (2×2) complex-valued matrices is a subspace of the span of $\{\mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3\}$ over $\mathbb{C}$.

(D) Any linear combination of $\mathbf{M}_1$, $\mathbf{M}_2$ and $\mathbf{M}_3$ can be diagonalized over $\mathbb{C}$.

(E) None of the above.

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十、Continued from Problem 九. Let $B = 3M_1 + 4M_2 + M_3$ and $C = B^4 - 4B^3 - 9B^2 + 27B + 11I_2$. Which of the following is/are true?

(A) $C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(B) $C = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$

(C) $C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(D) $C = \begin{bmatrix} 3 & 1 \\ 1 & 4 \end{bmatrix}$

(E) None of the above.

十一、Solve the first-order differential equation $x^2y'(x) + xy(x)\ln(y(x)) = xy(x)$. Which of the following statements is/are true?

(A) This is a homogeneous and linear differential equation

(B) $y(x) = 0$ is one particular solution

(C) $y(x) = \exp(1)$ is another particular solution

(D) $x = 0$ is also a solution

(E) None of the above.

十二、Continued from Problem 十一. Given the initial condition $y(a) = b$, which of the following statements is/are true?

(A) No solution if $a = 0$.

(B) A unique solution if $b = a > 0$.

(C) More than one solution if $b = \exp(1)$.

(D) A unique solution if $a \neq 0$ and $b > 0$.

(E) None of the above.

十三、The second-order linear differential equation $(1 - x^2)y''(x) + 2xy'(x) - 2y(x) = f(x)$, for $-1 < x < 1$. To find the homogeneous solution, i.e. $f(x) = 0$, given one solution $y_1(x) = x$, the other linearly independent solution $y_2(x)$ can be derived by setting $y_2(x) = v(x)y_1(x)$. Assuming $v(x)$ satisfies $v(1) = 2$ and $v(2) = \frac{5}{2}$, which of the following statements about $v(x)$ is/are true?

(A) $x(1 - x^2)v''(x) + 2v'(x) = 0$

(B) $x(x^2 - 1)v''(x) + 2v'(x) = 0$

(C) $v'(x) = \frac{x^2-1}{x^2}$

(D) $v'(x) = \frac{x^2}{x^2-1}$

(E) None of the above.

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十四、 Continued from Problem 十三, find a particular solution for $f(x) = 1 - x^2$ by the method of variation of parameters, i.e., $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$, where $y_1(x)$ and $y_2(x)$ are obtained from Problem 十三. Which of the following statements regarding $u_1(x)$ and $u_2(x)$ are true?

(A) $u_1(x) = \ln(1+x) - \ln(1-x) - x$

(B) $u_2(x) = \ln(1+x) + \ln(1-x)$

(C) $u_1(x) = x + \frac{x^3}{3}$

(D) $u_2(x) = -\frac{x^2}{2}$

(E) None of the above.

十五、 Solve the initial value problem of $(2x - x^2)y''(x) - 5(x-1)y'(x) - 3y(x) = 0$ with $y(1) = 0$ and $y'(1) = 1$ by power series of the form $y(x) = \sum_{n=0}^{\infty} c_n(x-1)^n$. Which of the following statements is/are true?

(A) $x = 0$ is a regular singular point.

(B) $x = 1$ is a regular singular point.

(C) The guaranteed radius of convergence is 2.

(D) The series converges if $0 < x < 2$.

(E) None of the above.

十六、 Continued from Problem 十五, which of the following statements regarding the recurrence relation as well as values of coefficients c_n is/are true?

(A) $c_{n+2} = \frac{n+2}{n+3}c_n$

(B) $c_8 = 0$

(C) $c_5 = \frac{8}{5}$

(D) $c_9 = \frac{63}{128}$

(E) None of the above.

十七、 Let $\underline{y}(x) = [y_1(x)\ y_2(x)]^\top$ and consider the following system of first-order differential equations

$$\underline{y}'(x) = \begin{bmatrix} 6 & -7 \\ 1 & -2 \end{bmatrix} \underline{y}(x) + \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Assuming $\underline{y}(0) = [1\ \ -1]^\top$, let $\underline{Y}(s) = [Y_1(s)\ Y_2(s)]^\top = \mathscr{L}\{\underline{y}(x)\}$. Which of the following statements is/are true?

(A) $Y_1(1) = \frac{5}{8}$.

(B) $Y_1(6) = \frac{85}{42}$.

(C) $Y_1(7) - Y_2(7) = \frac{11}{14}$.

(D) $\frac{Y_2(8)}{Y_1(8)} = 0$.

(E) None of the above.

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十八、 Continued from Problem 十七, which of the following statements regarding the solution $y(x)$ is/are true?

(A) $y_2'(0) = 6$.

(B) $y_1''(0) = 8$.

(C) $y_2''(0) = 3$.

(D) $\lim_{x\to-\infty}(y_1(x) - y_2(x)) = \frac{1}{4}$.

(E) None of the above.

十九、 Let $y(x)$ be a real-valued function satisfying the following second-order differential equation

$$y''(x) + y(x) = f(x)$$

Assume $y(0) = y'(0) = 0$ and $f(x) = \sum_{n\geq 0} u(x - n\pi)\sin(x - n\pi)$. Which of the following statements is/are true?

(A) $y(x) = \left(\mathscr{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right)\right\} \star \left(\sum_{n\geq 0}\delta(x - n\pi)\right)$.

(B) $\mathscr{L}^{-1}\left\{\frac{s^2}{(s^2+1)^2}\right\} = \frac{1}{2}[\sin(x) + x\cos(x)]u(x)$.

(C) Let $F(s) = \mathscr{L}\{f(x)\}$; then $F(1) = \frac{1}{2(1-e^{-\pi})}$.

(D) $y(x) = 0$ for all $x < -\pi$.

(E) None of the above.

二十、 Continued from Problem 十九, which of the following statements is/are true?

(A) $y(x)$ is a periodic function with period π.

(B) $y(x)$ is bounded for all $x > 0$.

(C) $y\left(\frac{\pi}{2}\right) = \frac{1}{2}$.

(D) $y'\left(\frac{\pi}{2}\right) = \frac{\pi}{4}$.

(E) None of the above.

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