

※請在答案卷內作答



Note: Detailed derivations are required to obtain a full score for Problem 2 to Problem 4.

1. (15%) Among the 10 statements below, only 5 are true and the other 5 are false. Find out which 5 are true. (You are not obligated to give explanations, but **will get zero point if listing more than 5 of them**).

 - (a) Let V be a vector space and $S \subseteq V$ be a subset. Then, $\text{span}(S)$ is the intersection of all subspaces of V that contain S .
 - (b) Let $T : V \rightarrow W$ be a linear transformation. Let $S = \{v_1, v_2, \dots, v_n\}$ be a subset of V . If S is linearly dependent, its image $T(S)$ is also linearly dependent.
 - (c) The basis of any vector space uniquely exists.
 - (d) Let $T : V \rightarrow W$ be a linear transformation. If T is invertible, then $\dim(V) = \dim(W)$.
 - (e) Let $A \in M_{m \times n}(\mathbb{R})$ be an arbitrary matrix. If $m < n$, then $\text{rank}(A) > \text{rank}(A^t)$.
 - (f) Assume that $A \in M_{m \times n}(\mathbb{R})$ and $b \in M_{m \times 1}(\mathbb{R})$. Let x_1 and x_2 be two column vectors in \mathbb{R}^n . If $x_1 \neq x_2$ and $Ax_1 = b = Ax_2$, then the system of linear equations $Ax = b$ has infinitely many solutions.
 - (g) Let A and B be square matrices of the same size. If $AB = O$, then $R(L_B) \supseteq N(L_A)$. (Remarks: L_A and L_B denote the linear transformation of matrix multiplication from the left.)
 - (h) Let A and B be square matrices of the same size. If $AB = A$, then $B = I$.
 - (i) Let A be a square matrix and $r \in \mathbb{R}$. Then, $\det(rA) = r \det(A)$.
 - (j) Assume that $A \in M_{3 \times 3}(\mathbb{C})$ and $A^t A = -I$. Then, the entries in A cannot all be real numbers.

2. (10%) Define a linear transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ by $T((1, 0, 0)) = (0, 1, 0)$, $T((0, 1, 0)) = (0, 0, 1)$, and $T((0, 0, 1)) = (1, 0, 0)$.
 - (a) (5%) Find a vector $u = (u_x, u_y, u_z)$ such that $T(u) = u$ and $\sqrt{u_x^2 + u_y^2 + u_z^2} = 1$.
 - (b) (5%) Is $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ one-to-one and onto? Why or why not?

3. (15%) Let W_1, \dots, W_k be subspaces of a vector space V . The direct sum V of W_1, \dots, W_k is defined if the following two conditions hold.

$$V = \sum_{i=1}^k W_i \quad \text{and} \quad W_j \cap \sum_{i \neq j} W_i = \{0\} \quad \forall j (1 \leq j \leq k)$$

If the two conditions hold, then V is denoted by $V = W_1 \oplus \dots \oplus W_k$. Prove or disprove (by providing a counterexample) of the following statements.



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(a) (7%) If $V = W_1 \oplus \dots \oplus W_k$. Then, for any distinct i and j , W_i and W_j intersect at exactly the zero vector.

(b) (8%) If $V = \sum_{i=1}^n W_i$, and W_i and W_j intersect at exactly the zero vector for any distinct i, j ($1 \leq i, j \leq k$). Then V is the direct sum of W_1, \dots, W_k .

4. (10%) Let V be a finite-dimensional complex inner product space and $T : V \rightarrow V$ be a linear operator. T is normal if and only if $TT^* = T^*T$, where T^* is the adjoint of T . Moreover, T is **nilpotent** if there exists $n \in \mathbb{N}$ such that T^n is the zero operator. Prove the following statement. If T is both normal and nilpotent, then T is the zero operator itself.

5 (5%)

Solve $y'' + 5y' + 4y = e^{-x}$

6. (5%)

Solve the following differential equation with the initial conditions by using Laplace transform.

$$y'' + y = \delta(t - \pi)$$

$$y(0) = 0$$

$$y'(0) = 0$$

7. (5%) $y'' + (1 + \lambda)y = 0$; $y(0) = y(\pi) = 0$

Find the eigenvalues and eigenfunctions.

8. (5%)

$$f(t) = \begin{cases} e^{-\alpha t}, & t \geq 0 \\ 0, & t < 0 \end{cases} \quad (\alpha > 0)$$

Find the Fourier Transform of $f(t)$.

9. (5%)

$$f(t) = \begin{cases} -1, & -\pi < t < 0 \\ 1, & 0 < t < \pi \end{cases}$$

$$f(t + n2\pi) = f(t)$$

Find the Fourier Series of $f(t)$.

注意：背面有試題

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10. (5%)

Find u and v such that $f(z) = u(x, y) + iv(x, y)$;
determine where f is differentiable and where f is not.

$$f(z) = |z^2|$$

11. (5%)

Find the principle value of $(i)^{1-2i}$

12. (5%)

Find the residue of $f(z) = \frac{\sin z}{z^4(z^2+i)}$ at $z = 0$.

13. (5%)

Consider the complex series $f(z) = \sum_{n=0}^{\infty} \frac{z^n}{4^{n+1}} + \sum_{n=1}^{\infty} \frac{1}{z^n}$.

Find the region of convergence.

14. (5%)

$$\oint_C e^{3/z} dz$$

where $C: |z-i|=2$, clockwise

