Part I: True (T) or False (F). You do not need to justify your answer. (30%; 3% each.)

1. The symbol $0.\overline{9} = 0.999999\ldots$ means there are infinitely many 9 after the decimal point, then $0.\overline{9} = 1$.

2. If $f'(x)$ is continuous on $[0,\infty)$ and $\lim_{x\to\infty} f(x) = 0$, then $\int_0^\infty x f'(x)dx + \int_0^\infty f(x)\mathrm{d}x = 0$.

3. There exists a power series $\sum_{n=0}^{\infty} a_n x^n$ which is convergent on $(-1,2)$ and divergent on $(\infty,-1] \cup [2,\infty)$.

4. Student A knew the formula: $2\sin x\cos y = \sin(x+y) + \sin(x-y)$. He has an idea that if we view x as a fixed number and view y as a variable, then we can differentiate this formula with respect to y on both sides to get
$$-2\sin x \sin y = \cos(x+y) - \cos(x-y).$$
Is this process correct and is this new formula correct as well?

5. Consider the limit $\lim_{n\to\infty}\left(\frac{1}{n^2}+\frac{2}{n^2}+\cdots+\frac{n}{n^2}\right)$. Student A gives the following argument: Since $\lim_{n\to\infty}\frac{k}{n^2} = 0$ for every $k = 1,2,\ldots,n$, by the Limit Sum Law, we have
$$\lim_{n\to\infty}\left(\frac{1}{n^2}+\frac{2}{n^2}+\cdots+\frac{n}{n^2}\right) = \lim_{n\to\infty}\frac{1}{n^2}+\lim_{n\to\infty}\frac{2}{n^2}+\cdots+\lim_{n\to\infty}\frac{n}{n^2} = 0+0+\cdots+0 = 0.$$
Is this argument correct?

6. Let $f(x) = \begin{cases} x^2 & \text{if } x<1 \\ 2x & \text{if } x \geq 1 \end{cases}$. Find $f'(x)$. Student A gives the following argument:
   - If $x<1$, then $f'(x) = 2x$.
   - If $x>1$, then $f'(x) = 2$.
   - If $x=1$, then $\lim_{x\to1^-} f'(x) = \lim_{x\to1^+} f'(x) = 2$. So $f'(1) = 2$.

   Thus, we know that $f'(x) = \begin{cases} 2x & \text{if } x<1 \\ 2 & \text{if } x \geq 1 \end{cases}$. Is this argument correct?

7. Student A has an idea to find the definite integral $\int_0^{\frac{\pi}{2}} \sin^2\theta \mathrm{d}\theta$. First, from the fact $\sin^2\theta + \cos^2\theta = 1$, it implies $\int_0^{\frac{\pi}{2}}(\sin^2\theta+\cos^2\theta)\mathrm{d}\theta = \frac{\pi}{2}$. Next, from the fact $\cos\theta = \sin\left(\frac{\pi}{2}-\theta\right)$, by the Substitution Rule and the concept of dummy variable, we have $\int_0^{\frac{\pi}{2}}\cos^2\theta\,\mathrm{d}\theta = \int_0^{\frac{\pi}{2}}\sin^2\left(\frac{\pi}{2}-\theta\right)\mathrm{d}\theta = -\int_{\frac{\pi}{2}}^{0}\sin^2\varphi\,\mathrm{d}\varphi = \int_0^{\frac{\pi}{2}}\sin^2\theta\,\mathrm{d}\theta$. Hence $\int_0^{\frac{\pi}{2}}\sin^2\theta\mathrm{d}\theta = \frac{\pi}{4}$.
Is this idea correct?

8. Consider one loop of the four-leaved rose curve $r = \cos 2\theta, -\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$, in polar equation. The enclosed area formula of this curve can be written as
$$\text{Enclosed Area} = \int_{\frac{\pi}{4}}^{-\frac{\pi}{4}} y(\theta)\mathrm{d}x(\theta) = \int_{\frac{\pi}{4}}^{-\frac{\pi}{4}} \cos 2\theta \sin\theta \mathrm{d}(\cos 2\theta \cos\theta).$$

9. Consider an infinite sequence $\{a_n\}_{n=1}^{\infty}$, where $a_{n+1} = (-1)a_n + \frac{1}{n}$ and $a_1 = -1$. Student A gives the following argument: Let $\lim_{n\to\infty} a_n = L$, then
$$\lim_{n\to\infty} a_{n+1} = (-1)\lim_{n\to\infty} a_n + \lim_{n\to\infty}\frac{1}{n} \Rightarrow L = -L + 0 \Rightarrow L = 0.$$
Is this argument correct?

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10. Student A computed the indefinite integral $\int 2\sin\theta\cos\theta d\theta = 2\int \sin\theta d\sin\theta = \sin^2\theta + C$.
Student B computed the indefinite integral $\int 2\sin\theta\cos\theta d\theta = -2\int \cos\theta d\cos\theta = -\cos^2\theta + C$.
Student C computed the indefinite integral $\int 2\sin\theta\cos\theta d\theta = \int \sin 2\theta d\theta = -\frac{1}{2}\cos 2\theta + C$.
All students computed the indefinite integral correctly.

PART II: Answer each question. (20%; 4% each.)

11. $\frac{d}{dx}\left(x^2 e^{\sin(x^3-x)}\right) = \underline{(11)}$.

12. Suppose that $f(x)$ is continuous on $(0,\infty)$ and $F(x) = \int_{\frac{1}{x}}^{\ln x}(x - f(t))dt$, then $F'(x) = \underline{(12)}$.

13. Reverse the order of the following integration (integrating first with respect to x and then y):

$$\int_0^1 \int_0^{\sqrt{x}} f(x,y)dydx + \int_1^2 \int_0^{2-x} f(x,y)dydx = \underline{(13)}.$$

14. Find the limit $\lim_{x\to\infty} \frac{\ln(1+4e^{2x})}{\sqrt{2+3x^2}} = \underline{(14)}$.

15. Let $f(x,y) = x^2y + 2xy - 3$ and $P = (1,1)$. Find the direction (vector v with unit length) in which $f(x,y)$ decreases most rapidly at P. $v = \underline{(15)}$.

PART III: Solve the following problems. **You need to write down complete arguments.**

16. Consider the ellipse $E: 13x^2 - 10xy + 13y^2 = 72$.
   (a) (4%) Find an equation of the tangent line to the ellipse E at the point $P\left(0, \frac{6}{13}\sqrt{26}\right)$.
   (b) (8%) Write a polar equation of the ellipse E. Find the area of the ellipse E by integrating this polar equation. You may need to use the substitution $\varphi = 2\theta$ and $t = \tan\left(\frac{\varphi}{2}\right)$, where θ is the polar angle.
   (c) (8%) Find the area of the ellipse E by changing of variables in multiple integrals. You need to compute the Jacobian of the transformation.
   (d) (10%) Find the area of the ellipse E by the Lagrange multiplier method. Here is the idea to realize the method: Notice that the center of the ellipse E is $O(0,0)$. We can find the maximum distance and minimum distance from the point $P(x,y)$ on the ellipse E to the center $O(0,0)$. They will correspond to the length of the semi-major a and the semi-minor axis b, respectively. The area of the ellipse is $ab\pi$.

17. (a) (6%) Find the Maclaurin series for $f(x) = \sin^2 x$. Hint: Half-angle formula $\sin^2 x = \frac{1-\cos 2x}{2}$.
   (b) (4%) Find $f^{(106)}(0)$ and $f^{(2017)}(0)$.

18. (10%) Solve the initial-value problem:

$$(x^2+1)y' + (x+1)^2 y = x^2 + 1, \quad y(0) = 1.$$

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