1. (20%)Please provide the general solutions of the following differential equations:

(a) (10%) $y''+10y'+25y=0$

(b) (10%) $3y'+y=(0-2x)y^4$

2. (15%) Lapalce Transform

(a) (5%) Find the inverse Laplace transform of the following:

$$F(s)=\frac{32}{s(s+4)(s+8)}$$

(b) (10%) Solve the following system using **Laplace Transform**:

$$x''-2x'+3y'+2y=4$$
$$2y'-x'+3y=0$$
$$x(0)=x'(0)=y(0)=0$$

3. (15%) Vector

(a) (7%) Consider $\varphi(x,y,z)=z-\sqrt{x^2+y^2}$. The level surface $\varphi(x,y,z)=0$ is the cone $z=\sqrt{x^2+y^2}$. Find a normal vector and the tangent plane at $(1, 1, \sqrt{2})$.

(b) (8%) Let C be the curve consisting of the quarter-circle $x^2+y^2=1$ in the x, y-plane from (1,0) to (0,1), then the horizontal line segment from (0,1) to (2,1). Let $\vec{F}(x,y)=4x\vec{i}$, Compute $\int_C \vec{F}\cdot d\vec{R}$.

4. (20%) Matrices

(a) (4%) Consider a linear system

$x_1 + 2x_2 + 5x_3 = 0$
$2x_1 + 3x_2 + 8x_3 = -5$
$-x_1 + x_2 + 2x_3 = 4$

The equations of the system can be written in the form of matrix as $\boldsymbol{AX}=\boldsymbol{B}$. Find the rank of the augmented matrix of the system $(\boldsymbol{A}|\boldsymbol{B})$

(b) (5%) Find the inverse of the coefficient matrix $\boldsymbol{A}$ and use the inverse matrix $\boldsymbol{A}^{-1}$ to solve the equations.

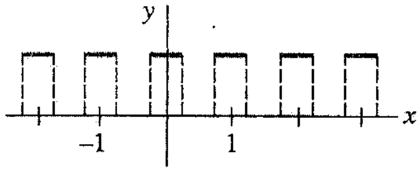
(c) (7%)Determine the eigenvalues and eigenvectors of the following matrix **M**

$$\mathrm{M}=\begin{pmatrix}1 & 0 & 1\\ 0 & -1 & 3\\ 0 & 0 & 2\end{pmatrix}$$

(d) (4%) Diagonalize matrix **M**

5. (10%) The periodic square wave or periodic pulse is shown in Figure. The wave is the periodic extension of the function f:

$$f(x)=\begin{cases} 0, & -\tfrac{1}{2}<x<-\tfrac{1}{4} \\ 1, & -\tfrac{1}{4}<x<\tfrac{1}{4} \\ 0, & \tfrac{1}{4}<x<\tfrac{1}{2} \end{cases}$$



(a) (5%) Please explain the Fourier series of the function f

(b) (5%) Find the frequency spectrum of the function f

6. (20%) Solve the equation $u(x, t)$ subject to the given conditions.

(a) (10%)

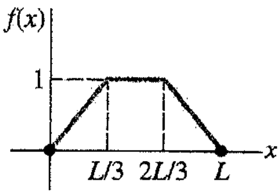
Wave equation: $k\dfrac{\partial^2 u}{\partial x^2}=\dfrac{\partial u}{\partial t}$, $\quad 0<x<L, \quad t>0$

Conditions:

$u(0,t)=0, \quad u(L,t)=0$

$u(x,0)$ as specified in Figure.

$\left.\dfrac{\partial u}{\partial t}\right|_{t=0}=0$



(b) (10%) The transverse displacement $u(x, t)$ of a vibrating beam of length L is determined from a fourth-order partial differential equation

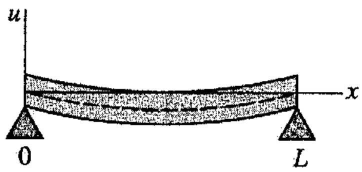
$$a^2\frac{\partial^4 u}{\partial x^4}+\frac{\partial^2 u}{\partial t^2}=0, \qquad 0<x<L, \qquad t>0.$$

If the beam is simply supported, as shown in Figure, the boundary and initial conditions are

$u(0,t)=0, \quad u(L,t)=0, \quad t>0$

$\left.\dfrac{\partial^2 u}{\partial x^2}\right|_{x=0}=0, \quad \left.\dfrac{\partial^2 u}{\partial x^2}\right|_{x=L}=0, \quad t>0$

$u(x,0)=f(x), \quad \left.\dfrac{\partial u}{\partial t}\right|_{t=0}=g(x), \quad 0<x<L$



Hint: For convenience use λ^4 instead of λ^2 when separating variables.