1. (8%) A fair coin is tossed until the same result occurs twice. Give the probability of an odd number of tosses.

2. (12%) If $X|Y \sim$ binomial $(Y,\ p)$, $Y|\Lambda \sim$ Poisson (Λ), and $\Lambda \sim$ exponential (β), i.e. $f_\Lambda(\lambda) = \frac{1}{\beta}e^{-\lambda/\beta}$.

   (a) (7%) Find the pmf of Y and the name of this pmf.

   (b) (5%) Compute $E(X)$.

3. (12%) Consider the bivariate negative binomial distribution with pmf

$$P(X=x, Y=y) = \frac{(x+y+k-1)!}{x!y!(k-1)!} p_1^x p_2^y (1-p_1-p_2)^k,$$

   where $x,\ y = 0, 1, 2, \cdots$; $k \geq 1$ is an integer; $p_1,\ p_2 \in (0,1)$; and $p_1 + p_2 < 1$.

   (a) (5%) Find the pmf of $U = X + Y$.

   (b) (7%) Find the moment generating function of U.

4. (18%) Let $X_1, \cdots,\ X_n$ be independent $N(0,\ \theta)$.

   (a) (4%) Find the MLE of θ^2.

   (b) (6%) Find the MVUE of θ^2.

   (c) (5%) Find a UMP test for testing $H_0 : \theta \leq \theta_0$ vs. $H_A : \theta > \theta_0$.

   (d) (3%) For testing $H_0 : \theta \leq 3$ vs. $H_A : \theta > 3$, if a sample of size n is 10 with $\sum x_i^2 = 54$, find the P-value for this test referring to the following values: $P(\chi_9^2 > 6) = 0.7399$, $P(\chi_{10}^2 > 6) = 0.8153$, $P(\chi_9^2 > 18) = 0.03517$, $P(\chi_{10}^2 > 18) = 0.05496$.

5. (15%) X is said to have a Maxwell distribution if its density function is given by:

$$f(x|\theta) = (2/\pi)^{1/2}\theta^{3/2}x^2 exp(-x^2\theta/2),\ x > 0.$$

Let $X_1,X_2\cdots,X_n$ be iid random samples from a Maxwell distribution with parameter θ. You can use the fact that $U = \sum_{i=1}^n X_i^2$ is a complete sufficient statistic for this family.

(a) (3%) Find the MLE of θ.

(b) (5%) Show that θU has a $\chi^2(3n)$ distribution.

(c) (7%) Find the $UMVUE$ of θ.

6. (27%)Let $X_1,\ X_2,\ldots,\ X_n$ be a random sample of size n from the distribution $U[0,\theta]$.

$$f(x) = \begin{cases} \frac{1}{\theta} & \text{if } 0 \le x \le \theta, \\ 0 & \text{elsewhere} \end{cases}$$

(a) (6%) Find the MVUE of θ.

(b) (4%) Find the variance of MVUE of θ.

(c) (4%) Compare the results in (b) with the Rao-Cramer's lower bound.

(d) (7%) Let $X_{(1)},\cdots,X_{(n)}$ denote the order statistics. The range is defined as $R = X_{(n)} - X_{(1)}$ and the mid-range is defined by $V = (X_{(1)} + X_{(n)})/2$. Find the joint pdf of $X_{(1)}$ and $X_{(n)}$.

(e) (3%) Based on the results in (d), find the marginal pdf of R.

(f) (3%) If now $\theta = 1$, based on the results in (e), find $E(R)$.

7. (8%) A scatter plot of y versus x for 30 subjects reveals no isolated cases and suggests that the relationship between y and x is linear. Computations yield, given the following information.

- $\bar{x} = 60$, $\sum_{i=1}^{30}(x_i - \bar{x})^2 = 400$, $\bar{y} = 100$, $\sum_{i=1}^{30}(y_i - \bar{y})^2 = 900$, and $r = -0.6$.
- One of the subjects, Ann has $x = 50$ and $y = 114$.

Answer the following questions.

(a) (5%) Obtain the equation of the regression line.

(b) (3%) Compute Ann's residual.