This exam has 9 questions, for a total of 100 points.

1. (a) (5 points) Find the absolute minimum and maximum values of $f(t) = t^3 - 3t^2 + 1$ on the interval $[-2, 2]$.
   (b) (5 points) Show that $e^{-x} > 1 - x$ for all $x < 0$.
2. (a) (5 points) Let
$$y = \int_{\sin x}^{\cos x} \frac{u}{\sqrt{1-u^2}}\, du.$$
Find $\frac{dy}{dx}$.
   (b) (5 points) Evaluate $\lim_{x\to+\infty}(1 + x^{-1})^{-x}$.
3. Determine whether the following statement is true or false and give reasons for your answer.
   (a) (5 points) If $c_n > 0$ and $\sum c_n$ converges, then $\sum c_n^4$ converges.
   (b) (5 points) If $\lim_{n\to\infty} a_n = 0$, then $\sum a_n$ converges.
4. (10 points) Use Newton's method to approximate a zero of $h(x) = x^2 - 2$ with initial guess $x_1 = 1$. Continue the iteration until two successive approximations differ by less than 0.01.
5. Let
$$f(x,y) = \begin{cases} \frac{x^3y - xy^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$
   (a) (5 points) Find $f_x(x,y)$ and $f_y(x,y)$ if $(x,y) \neq (0,0)$.
   (b) (10 points) Show that $f_{xy}(0,0) = -1$ and $f_{yx}(0,0) = 1$.
6. Let $\mathbf{X}$ be an $n \times p$ matrix with $n > p$. Suppose $\mathbf{X}^t$ is its transpose matrix and $\mathbf{X}^t\mathbf{X}$ is invertible. Denote $\mathbf{P} = \mathbf{X}(\mathbf{X}^t\mathbf{X})^{-1}\mathbf{X}^t$.
   (a) (5 points) Find the trace of $\mathbf{P}$.
   (b) (5 points) Find the determine of $\mathbf{P}$.
7. (a) (7 points) Reduce the equation $3x^2 + 4xy + 3y^2$ to a sum of square by finding the eigenvectors of corresponding $\mathbf{A}$.
   (b) (5 points) Find an orthogonal matrix $\mathbf{B}$ such that $\mathbf{B}^{-1}\mathbf{A}\mathbf{B}$ is diagonal.
   (c) (5 points) Find $\det(\mathbf{A}^{99})$.
8. (8 points) Let $\mathbf{P}$ be a solution of the linear equations system $\mathbf{AX} = \mathbf{b}$ and $\mathbf{N}$ be a solution to $\mathbf{AX} = \mathbf{0}$. Show that $\mathbf{P} + \mathbf{N}$ is a solution of $\mathbf{AX} = \mathbf{b}$.
9. (10 points) Let
$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$
be three vectors in $\mathcal{R}^3$. If
$$\det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \neq 0,$$
show that the three vectors are linearly independent.