For the following problems, $\mathbb{R}$ denotes the field of real numbers. A real vector space is a vector space over $\mathbb{R}$.

(1) **(12 Points)** Let $V = \{(a, b, c, d) \in \mathbb{R}^4 : a-2b+3c-4d = 0\}$. Find the dimension of V as a real vector space by finding a basis for it.

(2) **(20 Points)** Let $S = \{1, 2, 3\}$ and let V be the set of all the functions from S to $\mathbb{R}$. Define

$$(f+g)(x) = f(x) + g(x), \qquad (af)(x) = af(x), \qquad \text{where } x \in S,$$

for all $f, g \in V$ and $a \in \mathbb{R}$.

(a) Show that V is a real vector space.

(b) Define the functions χ_1, χ_2 and χ_3 in V as follows:

$$\begin{array}{ccc} S & \xrightarrow{\chi_1} & \mathbb{R}, \\ 1 & \longmapsto & 1 \\ 2 & \longmapsto & 0 \\ 3 & \longmapsto & 0 \end{array} \qquad \begin{array}{ccc} S & \xrightarrow{\chi_2} & \mathbb{R}, \\ 1 & \longmapsto & 0 \\ 2 & \longmapsto & 1 \\ 3 & \longmapsto & 0 \end{array} \qquad \begin{array}{ccc} S & \xrightarrow{\chi_3} & \mathbb{R}. \\ 1 & \longmapsto & 0 \\ 2 & \longmapsto & 0 \\ 3 & \longmapsto & 1 \end{array}$$

Show that $\{\chi_1, \chi_2, \chi_3\}$ is a basis for V over $\mathbb{R}$.

(c) Suppose given

$$\begin{array}{ccc} S & \xrightarrow{f} & \mathbb{R}, \\ 1 & \longmapsto & 1 \\ 2 & \longmapsto & 0 \\ 3 & \longmapsto & 0 \end{array} \qquad \begin{array}{ccc} S & \xrightarrow{g} & \mathbb{R}, \\ 1 & \longmapsto & 1 \\ 2 & \longmapsto & 2 \\ 3 & \longmapsto & 0 \end{array} \qquad \begin{array}{ccc} S & \xrightarrow{h} & \mathbb{R}. \\ 1 & \longmapsto & 1 \\ 2 & \longmapsto & 2 \\ 3 & \longmapsto & 1 \end{array}$$

Is the set $\{f, g, h\}$ a basis for V over $\mathbb{R}$?

(3) **(15 Points)** Let $\{v_1, v_2, \ldots, v_n\}$ be a basis for the vector space V over $\mathbb{R}$. Let $w_1, w_2, \ldots, w_n$ also be elements in V. Let $T\colon V \longrightarrow V$ be the linear transformation sending v_i to w_i for each i. Show that T is an isomorphism if and only if $\{w_1, w_2, \ldots, w_n\}$ is a basis for V over $\mathbb{R}$.

(4) **(20 Points)** Let $a_1, a_2, \ldots, a_n \in \mathbb{R}$.

(a) Show that

$$\det \begin{pmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} \end{pmatrix} = \prod_{1 \le i < j \le n} (a_j - a_i).$$

(b) For $i = 1, 2, \ldots, n$, let $v_i = (1, a_i, a_i^2, \ldots, a_i^{n-1})$ be vectors in $\mathbb{R}^n$. Give a sufficient and necessary condition for $v_1, v_2, \ldots, v_n$ to be linear independent over $\mathbb{R}$.

(5) **(18 Points)** Find the Jordan form of

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

over $\mathbb{R}$.

(6) **(15 Points)** Is it possible to find a 4×4 symmetric matrix with real entries such that its characteristic polynomial is $x^4 - 1$? Give your reasoning.