1. a. (5 pts) Let $\beta \in \mathbb{R}$ and $A$ be a nonempty subset of $\mathbb{R}$. Write down the definition of

$\beta$ is the supremum of $A$.

b. (10 pts) Suppose that $f : [a, b] \mapsto \mathbb{R}$ is an increasing continuous function. Show that $\sup f(E) = f(\sup E)$, where $E$ is a nonempty subset of $[a, b]$, and $\sup E$ and $\sup f(E)$ represent the supremum of $E$ and $f(E)$, respectively.

2. (10 pts) Suppose that $f : [a, b] \mapsto \mathbb{R}$ is continuously differentiable and $1-1$ on $[a, b]$. Show that

$$\int_a^b f(x)\,dx + \int_{f(a)}^{f(b)} f^{-1}(\xi)\,d\xi = bf(b) - af(a).$$

3. Let $f : [a, b] \mapsto \mathbb{R}$ be a function. Define

$$V(f, P) = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$$

for each partition $P = \{x_0, x_1, \cdots, x_n\}$ of $[a, b]$. The variation of $f$ is defined by

$$Var(f) = \sup\{V(f, P) \mid P \text{ is a partition of } [a, b]\}.$$

a. (6 pts) Show that if $Var(f) < \infty$, then $f$ is a bounded function on $[a, b]$.

b. (9 pts) Show that the variation of $f(x) = \begin{cases} \sin\frac{1}{x} & \text{if } x \in (0, \frac{2}{\pi}] \\ 0 & \text{if } x = 0 \end{cases}$ is not finite.

c. (10 pts) Suppose $f'$ exists and is integrable on $[a, b]$. Prove that

$$Var(f) = \int_a^b |f'(x)|\,dx.$$

4. a. (10 pts) Let $\alpha \in (0, 1]$, and $\delta \geqslant -1$. Prove that $(1+\delta)^\alpha \leqslant 1 + \alpha\delta$.

b. (10 pts) Show that

$$\lim_{n\to\infty} \int_1^5 \left(1 + \frac{x}{n}\right)^n e^{-x}\,dx = 4.$$

5. a. (10 pts) Is the function $f(x,y) = \sqrt{|xy|}$ differentiable at $(0, 0)$ ? Prove your assertion.

b. (10 pts) Suppose $f : \mathbb{R}^2 \mapsto \mathbb{R}$ satisfies

$$|f(x_1,y_1) - f(x_2,y_2)| \leqslant 2011((x_2-x_1)^2 + (y_2-y_1)^2)$$

for all $(x_1,y_1), (x_2,y_2) \in \mathbb{R}^2$. Is $f$ a differentiable function in $\mathbb{R}^2$ ? Prove your assertion.

6. (10 pts) Find extrema of $f(x,y) = x^2 + xy + 4y^2$ on $\Omega$, where $\Omega$ is the region bounded by the triangle with vertices $(1,0)$, $(1,2)$, and $(3,0)$.