國立中山大學100學年度碩士班招生考試試題

科目：線性代數【通訊所碩士班甲組】

單選題 (6x5%=30%):

1. Consider a 3 × 3 matrix

$$A = \begin{bmatrix} 2 & -i & 1+i \\ i & 1 & 0 \\ 1-i & 0 & 1 \end{bmatrix},$$

[i] **A** is an Hermitian matrix

[ii] **A** is positive definite

[iii] The determinant of **A** is 1

[iv] The eigenvalues of **A** are 2, 1 and 1

[v] The trace of **A** is 4

Which statements are correct?

(a) i、ii、iv (b) i、iv、v (c) ii、iv、v

(d) i、ii、iii、v (e) i、v

2. Let $\mathrm{B} \in \mathcal{R}^{n\times n}$ be an orthogonal matrix :

[i] The eigenvalues of **B** are 1, 0, or -1

[ii] The determinant of **B** is 1

[iii] The columns of **B** form an orthonormal basis of R^n

[iv] For any $\mathrm{x} \in \mathcal{R}^{n\times 1}$, $\|\mathrm{x}\| = \|\mathrm{Bx}\|$

[v] For any $\mathrm{x}, \mathrm{y} \in \mathcal{R}^{n\times 1}$, $\langle \mathbf{x},\mathbf{y} \rangle = \langle \mathbf{Bx},\mathbf{By} \rangle$; where $\langle \mathbf{x},\mathbf{y} \rangle = \mathbf{y}^T\mathbf{x}$ is the inner product of vectors **x** and **y**.

Which statements are always correct?

(a) i、 iv、v (b) ii、iii、iv (c) iii、iv、v

(d) i、ii、v (e) i、ii、iii

3. In the following, which is NOT diagonalizable?

(a) $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ (e) $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

國立中山大學100學年度碩士班招生考試試題

科目：線性代數【通訊所碩士班甲組】

4. Consider three vectors in R^4: $[\ 1, 0, 1, 0\ ]^T$, $[\ 0, 1, -2, 1\ ]^T$, $[\ 1, -1, 0, 0\ ]^T$. Among the following vectors, which form an orthonormal basis of the subspace spanned by three vectors?

   [i] $\left[\ 1/\sqrt{2},\ \ 0,\ \ 1/\sqrt{2},\ \ 0\ \right]^T$

   [ii] $\left[\ 0,\ \ 1/\sqrt{6},\ \ -2/\sqrt{6},\ \ 1/\sqrt{6}\ \right]^T$

   [iii] $\left[\ 1/\sqrt{6},\ \ -2/\sqrt{6},\ \ -1/\sqrt{6},\ \ 0\ \right]^T$

   [iv] $\left[\ 1/\sqrt{2},\ \ -1/\sqrt{2},\ \ 0,\ \ 0\ \right]^T$

   [v] $\left[\ 1/2,\ \ 1/2,\ \ -1/2,\ \ 1/2\ \right]^T$

   (a) i、iii、v (b) i、ii、v (c) iii、iv、v

   (d) i、ii、iv (e) i、iii、iv

5. Given $n \times n$ matrices **A** and **B**, and there is an $n \times n$ invertible matrix **P** such that $\mathbf{B} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$.

   [i] **A** and **B** have the same trace

   [ii] **A** and **B** have the same eigenvectors

   [iii] **A** and **B** have the same determinant

   [iv] **A** and **B** have the same eigenvalues

   [v] **A** and **B** are simultaneously diagonalizable

   Among the above statements, which are not always true?

   (a) ii、iv、v (b) i、iv、v (c) i、iii、iv

   (d) ii、v (e) ii、iii

6. Let **A** be an $n \times n$ matrix with n real eigenvalues $\lambda_1 > \lambda_2 > \cdots > \lambda_n > 0$,

   [i] For any nonzero $n \times 1$ vectors **x**, $\lambda_n \le \frac{\mathbf{x}^H\mathbf{A}\mathbf{x}}{\|\mathbf{x}\|^2} \le \lambda_1$

   [ii] The determinant of $\mu\mathbf{A}^2$ is $\mu\lambda_1^2\lambda_2^2\cdots\lambda_n^2$

   [iii] The eigenvalues of matrix $(\mathbf{A}+\mathbf{I})^{-1}$ are the same as those of $\mathbf{A}^{-1}$

   [iv] The eigenvectors of matrix $(\mathbf{A}+\mathbf{I})^{-1}$ are the same as those of $\mathbf{A}^{-1}$

   [v] Given a nonzero $n \times 1$ vector **v** and for any nonzero $n \times 1$ vectors **x**, $\frac{|\mathbf{v}^T\mathbf{x}|^2}{\mathbf{x}^T\mathbf{A}\mathbf{x}} \le \mathbf{v}^T\mathbf{A}^{-1}\mathbf{v}$

   Among above statements, which are not always true?

   (a) ii、iii (b) iii、v (c) iii、iv、v

   (d) iii、iv (e) i、ii

國立中山大學100學年度碩士班招生考試試題

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計算證明題 (70%)

1. Given singular value decomposition of a matrix $\mathbf{H} \in \mathcal{C}^{m\times n}$ as $\mathbf{H} = \mathbf{U\Sigma V}^H$, where $\mathbf{U}$ and $\mathbf{V}$ are $m \times m$ and $n \times n$ unitary matrices, Σ is an $m \times n$ diagonal matrix composed of nonnegative singular values $\sigma_1, \cdots, \sigma_r, 0, \cdots, 0$, where $r = \text{rank}(\mathbf{H})$.
   (a) Find the eigenvalues of $\mathbf{HH}^H$. (5%)
   (b) Prove that (5%)
$$\sum_{i=1}^{r} \sigma_i^2 = \sum_{i=1}^{m}\sum_{j=1}^{n} |h_{i,j}|^2 .$$

2. Find the LU decomposition of the following matrix (10%):
$$\mathrm{A} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

3. Please use LU–decomposition to solve the following system of linear equations (10%):
$$\begin{cases} -X_1 + 2X_2 - X_3 = 2 \\ X_1 - 4X_2 + 6X_3 = -3 \\ -2X_1 + 6X_2 - 6X_3 = 8 \end{cases}$$

4. Let $B_0 = \left\{\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}\right\}$, $B_1 = \left\{\begin{bmatrix}2\\1\end{bmatrix}\begin{bmatrix}3\\-1\end{bmatrix}\right\}$ and $B_2 = \left\{\begin{bmatrix}2\\3\end{bmatrix}\begin{bmatrix}-4\\1\end{bmatrix}\right\}$ be three bases in R^2.
   Let $X = \begin{bmatrix}5\\2\end{bmatrix}$ in B_0
   (a) Write X in term of the vector in B_2. (5%)
   (b) Find the transformation matrix that converts a vector from in terms of Base B_1 to Base B_2. (5%)

5. Consider the vector space R^3 with Euclidean inner product. Apply the Gram-Schmidt process to transform the basis vector $u_1 = (0,1,0)$, $u_2 = (1,1,1)$ and $u_3 = (1,1,2)$ into an orthogonal basis $\{v_1, v_2, v_3\}$; then normalize the orthogonal basis vectors to obtain an orthonomal basis $\{q_1, q_2, q_3\}$ (10%).

國立中山大學100學年度碩士班招生考試試題

6. Find a singular value decomposition of $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 1 & -1 \end{bmatrix}$. (10%)

7. Consider the vector space P_3 of polynomials of degree less than 3, and the ordered basis $B = \{x^2, x, 1\}$ for P_3. Let T: $P_3 \rightarrow P_3$ be the linear transformation such that

$$T(ax^2 + bx + c) = (a - c)x^2 - bx + 2c$$

Find the eigenvalues and the eigenvectors for the linear transformation T. (10%)