系所組別： 數學系應用數學

考試科目： 高等微積分

考試日期：0219，節次：3

※ 考生請注意：本試題 ☐可 ☑不可 使用計算機

1. Let $\{x_k\}$ be a bounded sequence in $\mathbb{R}$, and let $Y_m = \sup\{x_k \mid k \geq m\}$.
   (a) (6 points) Show that $\lim_{m\to\infty} Y_m$ exists.
   (b) (6 points) The limit superior of the sequence $\{x_k\}$, denoted $\limsup_{k\to\infty} x_k$, is defined by $\limsup_{k\to\infty} x_k = \lim_{m\to\infty} Y_m$. Show that $\limsup_{k\to\infty} x_k = a$ if and only if for any $\epsilon > 0$, there are infinitely many k for which $x_k > a - \epsilon$ but only finitely many for which $x_k > a + \epsilon$.

2. Let $f(x,y) = \begin{cases} \dfrac{x\left(x^2 - y^2\right)}{x^2 + y^2}, & \text{if } (x,y) \neq (0,0), \\ 0 \quad , & \text{if } (x,y) = (0,0). \end{cases}$
   (a) (6 points) Show that f is continuous at $(0,0)$.
   (b) (6 points) For each unit vector $u \in \mathbb{R}^2$, show that the directional derivative of f at $(0,0)$ in the direction u exists, and compute it.
   (c) (4 points) Show that f is not differentiable at $(0,0)$.

3. (10 points) Let S be a compact subset of $\mathbb{R}^n$. Assume that $f : S \to \mathbb{R}$ is continuous, and $f(x) > 0$ for every $x \in S$. Show that there is a number $c > 0$ such that $f(x) \geq c$ for every $x \in S$.

4. (10 points) Find the 3rd-order Taylor polynomial of $f(x,y,z) = x^2 y^2 + z y$ based at $a = (1,2,1)$

5. (a) (4 points) Let $f(x,y) = \left(x + 2y,\ e^{2x} \sin y,\ x + \log(1 + y^2)\right)$, and $g : \mathbb{R}^2 \to \mathbb{R}^2$ be of class C^1 with $g(1,2) = (0,0)$, and $Dg(1,2) = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$. Compute $D(f \circ g)(1,2)$.
   (b) (6 points) Suppose that f is a homogeneous function of degree a on $\mathbb{R}^n$, i.e. $f(tx) = t^a f(x)$ for all $t > 0$ and $x \neq 0$. Show that $\displaystyle\sum_{j,k=1}^{n} x_j x_k \frac{\partial^2 f}{\partial x_j \partial x_k} = a(a-1)f$.

6. (12 points) Let $f(x,y) = \left(2x^2 + y^2\right)e^{-x^2-y^2}$. Find and classify the critical points of f.

7. Let $x_1, x_2, \ldots, x_n$ denote nonnegative numbers, and $c > 0$.
   (a) (8 points) Use Lagrange's method to find the maximum of the product $x_1 x_2 \cdots x_n$ subject to the constraint $x_1 + x_2 + \cdots + x_n = c$.
   (b) (6 points) Derive the inequality of geometric and arithmetic means, i.e. $\left(x_1 x_2 \cdots x_n\right)^{1/n} \leq \dfrac{x_1 + x_2 + \cdots + x_n}{n}$, and show that the equality holds if and only if the x_j's are all equal.

8. (a) (6 points) Evaluate $\displaystyle\int_0^2 \int_{y/2}^1 y e^{-x^3}\, dx dy$
   (b) (6 points) Let S be the region in the first quadrant bounded by the curves $xy = 1$, $xy = 3$, $x^2 - y^2 = 1$, and $x^2 - y^2 = 4$. Compute $\iint_S \left(x^2 + y^2\right) dA$.
   (c) (4 points) Let R be a regular region in $\mathbb{R}^3$ with piecewise smooth boundary, and let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field defined by $F(x,y,z) = x\mathrm{i} + y\mathrm{j} + z\mathrm{k}$. Show that the volume of R is $\dfrac{1}{3}\displaystyle\iint_{\partial R} F \cdot n\, dA$.