國立政治大學 105 學年度碩士班招生考試試題

考試科目	線性代數 81112、81162	所別	應用數學系	考試時間	2月28日(日) 第二節

Show all your work.

1. (20 pts) Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $T(a, b, c) = (a - b, 2c - a)$. Describe $T^{-1}(3, -2)$.

2. (a) (10 pts) Find an orthonormal basis for the subspace spanned by $\vec{x_1} = (2, 0, -1, 2)$, $\vec{x_2} = (0, 1, 1, -2)$ and $\vec{x_3} = (3, -1, 1, 0)$.

   (b) (10 pts) What is the projection of $(2, 5, 7, -3)$ in this space?

3. Let A denote the matrix

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 5 & 0 \\ -2 & 0 & 4 \end{bmatrix}$$

   (a) (5 pts) Find the eigenvalues of A.

   (b) (5 pts) Find an orthonormal basis of $\mathbb{R}^3$ consisting of eigenvectors for A.

   (c) (5 pts) Find a 3×3 orthogonal matrix S and 3×3 diagonal matrix D such that $A = SDS^T$.

   (d) (5 pts) For any integer k, write an explicit formula for A^k.

4. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation with the property that $T(T(\vec{x})) = T(\vec{x})$ for every vector $\vec{x} \in \mathbb{R}^n$.

   (a) (5 pts) Write V for the range of T. In other words, $V = \{T(\vec{x}) | \vec{x} \in \mathbb{R}^n\}$. If $\vec{x} \in V$, then what is $T(\vec{x})$?

   (b) (5 pts) If $\vec{x} \in \mathbb{R}^n$, then what is $T(\vec{x} - T(\vec{x}))$?

   (c) (5 pts) Let $\{\vec{v_1}, \vec{v_2}, \cdots, \vec{v_k}\}$ be a basis for V. Then we can add some more vectors $\vec{u_1}, \vec{u_2}, \cdots, \vec{u_\ell}$ to get a basis β for $\mathbb{R}^n$. Show that if you replace $\vec{u_1}$ with $\vec{u_1} - T(\vec{u_1})$, then you still have a basis.

   (d) (5 pts) In the same way, we can replace each $\vec{u_i}$ with $\vec{u_i} - T(\vec{u_i})$. What is the matrix of T with respect to the basis $\{\vec{v_1}, \vec{v_2}, \cdots, \vec{v_k}, \vec{u_1} - T(\vec{u_1}), \cdots, \vec{u_\ell} - T(\vec{u_\ell})\}$.

5. Show that

   (a) (10 pts) Let $A \in M_{n\times n}(F)$, and let B be a matrix obtained by adding a multiple of one row of A to another row of A. Then $\det(B) = \det(A)$.

   (b) (10 pts) Let $A \in M_{n\times n}(F)$ has rank less than n, then $\det(A) = 0$.

備註	一、作答於試題上者，不予計分。 二、試題請隨卷繳交。