國立交通大學100學年度碩士班考試入學試題

科目：微積分與線性代數(4081)　　考試日期:100年2月18日 第1節

系所班別：統計學研究所　　組別：統計所　　

【不可使用計算機】*作答前請先核對試題、答案卷(試卷)與准考證之所組別與考科是否相符！！

1. (10 points) Find the answers to the following questions:

   (1) (5 points) $\lim_{n\to\infty} \sqrt[n]{n!}$;

   (2) (5 points) $\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n}$.

2. (10 points) If $x = \int_0^y \frac{1}{\sqrt{1+t^2}}\,dt$, find $\frac{d^2y}{dx^2}$.

3. (10 points) Let $f(x) = ax^2/2$ with $a > 0$, and set

   $$g(y) = \sup_{-\infty<x<\infty}[xy - f(x)], \qquad y \in (-\infty, \infty).$$

   Find $g(y)$.

4. (10 points) Find the answers to the following questions:

   (1) (5 points) $\int_0^1 x\sqrt{1-x}\,dx$;

   (2) (5 points) $\lim_{x\to 0} \frac{1}{x^6}\int_0^{x^2} \frac{t^2}{t^6+1}\,dt$.

5. (10 points) Find the answers to the following questions:

   (1) (5 points) $\lim_{n\to\infty} \int_0^1 \frac{nx^{n-1}}{1+x}\,dx$;

   (2) (5 points) $\lim_{n\to\infty} \int_0^1 \frac{1+\sin(nx)}{1+x}\,dx$.

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6. Suppose that $\{\mathbf{x}_1, \cdots, \mathbf{x}_p\}$ is a set of linearly independent vectors in $\mathbf{R}^n$ with $n > p$. Define the $(n \times p)$ matrix $\mathbf{X} = [\mathbf{x}_1, \cdots, \mathbf{x}_p]$. Let $\mathcal{C}(\mathbf{X})$ represent the column space of $\mathbf{X}$.

   (a) (10 points) Show that $\mathbf{X}^T\mathbf{X}$ is invertible.

   (b) (10 points) For a vector $\mathbf{y} \in \mathbf{R}^n$, find the vector in $\mathcal{C}(\mathbf{X})$ that has the minimum distance with $\mathbf{y}$.

   (c) (10 points) Show that the vector you find in (b) is unique.

7. Following the above question, define $\bar{\mathbf{x}} = (1/p)\sum_{i=1}^{p} \mathbf{x}_i$ and $\mathbf{B} = [(\mathbf{x}_1 - \bar{\mathbf{x}}), \cdots, (\mathbf{x}_p - \bar{\mathbf{x}})]$. Please answer the following questions with appropriate explanation.

   (a) (5 points) What is the rank of $\mathcal{C}(\mathbf{B})$?

   (b) (5 points) Is $\mathbf{B}^T\mathbf{B}$ invertible?

   (c) (5 points) What is the rank of the orthogonal subspace of $\mathcal{C}(\mathbf{B})$?

   (d) (5 points) What is the number of nonzero eigenvalues of $\mathbf{B}$?