國立交通大學100學年度碩士班考試入學試題

科目：線性代數(4042)　　考試日期：100年2月18日 第2節

系所班別：應用數學系　　組別：應數系乙組　　第 1 頁,共 2 頁

【不可使用計算機】＊作答前請先核對試題、答案卷(試卷)與准考證之所組別與考科是否相符！！

1. Let $A = \begin{pmatrix} 3 & 4 \\ -1 & -1 \end{pmatrix}$ be a 2×2 real matrix.
   (a) (5 points) Compute A^{2011}.
   (b) (5 points) Compute A^{-19}.

2. (10 points) Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ be a 2×2 real matrix and suppose that A has an eigenvalue

$$\lambda = \max_{1 \le i \le 2} \left(\sum_{j=1}^{2} \frac{|a_{ij}| + |a_{ji}|}{2} \right).$$

   Prove that $A = A^T$, where A^T is the transpose of A.

3. Let $A = \begin{pmatrix} 0 & 2 & 2 \\ 0 & 3 & 3 \\ 0 & 4 & 4 \\ 3 & 0 & 6 \end{pmatrix}$ be a 4×3 real matrix.
   (a) (10 points) If possible, find a 3×4 matrix B and a 4×3 matrix C such that $BA = I_3$ and $AC = I_4$, where I_k is the $k \times k$ identity matrix for $k = 3, 4$. If not, explain why.
   (b) (10 points) Prove that $A^T A$ is diagonalizable with nonnegative eigenvalues $\lambda_1^2 \geq \lambda_2^2 \geq \lambda_3^2 \geq 0$, where A^T is the transpose of A and $\lambda_{j=1,2,3}$ are chosen to be nonnegative.
   (c) (10 points) Let e_1, e_2, e_3 be the canonical basis of $\mathbb{R}^3$. Prove or disprove that there is an orthonormal basis $\{v_j\}_{j=1}^4$ of $\mathbb{R}^4$ such that $Ae_j = \lambda_j v_j$ for $j = 1, 2, 3$. Here $\lambda_{j=1,2,3}$ is as above.

4. (10 points) Let $(V, \langle \cdot, \cdot \rangle_V)$ and $(W, \langle \cdot, \cdot \rangle_W)$ be two inner product spaces and $T : V \longrightarrow W$ be a linear transformation. We say T is an *isometry* if $\langle Tv_1, Tv_2 \rangle_W = \langle v_1, v_2 \rangle_V$.

   Now let $V = \mathbb{R}^3$ equipped the standard inner product structure and $W = P_2[x]$, the space of all polynomials with real coefficients of degree less than 3 equipped with the standard vector space structure and the inner product structure defined by

$$\int_0^1 p_1(x) p_2(x)\, dx \qquad \text{for all } p_1(x), p_2(x) \in P_2[x].$$

   Let $T : \mathbb{R}^3 \longrightarrow P_2[x]$ given by $T\left(\begin{pmatrix} a \\ b \\ c \end{pmatrix}\right) = \alpha + \beta x + \gamma x^2$ be an isometry. Express α, β and γ in terms of a, b and c explicitly.

5. Let $\mathrm{M}_n(\mathbb{R})$ be the collection of all $n \times n$ real matrices.
   (a) (8 points) Let $A, B \in \mathrm{M}_n(\mathbb{R})$. Prove that AB and BA have same eigenvalues.
   (b) (10 points) Prove that if $A, B \in \mathrm{M}_n(\mathbb{R})$ such that $AB = BA$, then A and B have a common eigenvector.

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(c) (8 points) Let $A \in \mathrm{M}_n(\mathbb{R})$ and let $p(t)$ is a polynomial function. Show that if A has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then $\operatorname{tr} p(A) = \sum_{i=1}^{n} p(\lambda_i)$, where $\operatorname{tr} B$ is the trace of a matrix $B \in \mathrm{M}_n(\mathbb{R})$.

(d) (4 points) We say $A \in \mathrm{M}_n(\mathbb{R})$ is *nilpotent* if, for some positive integer k, $A^k = 0$. Prove that if A is nilpotent, then $A^n = 0$.

(e) (10 points) Let $A, B \in \mathrm{M}_n(\mathbb{R})$ be nilpotent. Prove that if B has n distinct eigenvalues in $\mathbb{R}$ and $AB = BA$, then $A = 0$.